

COHOMOLOGY ALGEBRA OF PLANE CURVES, WEAK COMBINATORIAL TYPE, AND FORMALITY

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ABSTRACT. We determine an explicit presentation by generators and relations of the cohomology algebra $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ of the complement to an algebraic curve \mathcal{C} in the complex projective plane \mathbb{P}^2 via the study of log-resolution logarithmic forms on \mathbb{P}^2 . As a first consequence, we derive that $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ depends only on the following data: the number of irreducible components of \mathcal{C} together with their degrees and genera, the pairwise intersection numbers of distinct local branches at the singular points of \mathcal{C} , and, at each singular point, the incidence relations between the local branches and the global irreducible components passing through it. A further corollary is that the twisted cohomology jumping loci of $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ containing the trivial character also depend on the same data. We relate, in this context, the geometric and combinatorial properties of the curve \mathcal{C} through the notion of combinatorial pencil. This notion will be used to derive a combinatorial version of the Max-Noether Theorem. Finally, we find that the relations in the cohomology algebra are in fact satisfied, for appropriate choices of representatives, at the level of differential forms, concluding that $H^*(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ is a formal space.

1. INTRODUCTION

The combinatorial type $K(\mathcal{C})$ of a complex projective curve $\mathcal{C} \subset \mathbb{P}^2$ consists of the following list of data: the set of irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$ of \mathcal{C} together with their degrees $\bar{d} := (d_1, \dots, d_r)$, the set of singular points $\text{Sing}(\mathcal{C})$ of \mathcal{C} together with their topological types $\Sigma(\mathcal{C})$, and, at each singular point P , the correspondence ϕ_P between the local branches Δ_P and the global irreducible components.

The topological type $\Sigma(\mathcal{C})_P$ of the singular point P determines the pairwise intersection numbers between distinct branches in Δ_P . Also $\Sigma(\mathcal{C})$ together with the degrees \bar{d} determine the genera $\bar{g}(\mathcal{C}) = (g_1, \dots, g_r)$ of the irreducible components of \mathcal{C} . If we replace in $K(\mathcal{C})$ the set $\Sigma(\mathcal{C})$ by all these intersection numbers and \bar{g} , we obtain a new list $W_{\mathcal{C}} = (W, \bar{d}, \bar{g})$ which we call the weak combinatorial type of \mathcal{C} .

Note that the combinatorial type of \mathcal{C} determines the abstract topology of \mathcal{C} itself. That is not the case for the topology of the embedding $\mathcal{C} \subset \mathbb{P}^2$, as shown by Zariski's classical work where he showed that the fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is not determined by $K(\mathcal{C})$.

In this paper we will be concerned with the connection between the combinatorics of \mathcal{C} and the topology of its complement $S_{\mathcal{C}} = \mathbb{P}^2 \setminus \mathcal{C}$. The focus will be the cohomology algebra $H^*(S_{\mathcal{C}}, \mathbb{C})$ of \mathcal{C} with complex coefficients. It is not hard to see that the Betti numbers of $S_{\mathcal{C}}$ depend only on the number, degrees and genera of the irreducible components of \mathcal{C} .

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In a recent work by the first author it is proved that $H^*(S_{\mathcal{C}}, \mathbb{C})$ depends only on $W_{\mathcal{C}}$ in the case when \mathcal{C} is an arrangement of rational curves. Here we extend that result to arbitrary curves by showing that $H^*(S_{\mathcal{C}}, \mathbb{C})$ depends only on the weak combinatorial type $W_{\mathcal{C}}$.

The result follows from an explicit presentation of the cohomology algebra $H^*(S_{\mathcal{C}}, \mathbb{C})$ which is obtained by means of the Poincaré residue operators of [14, 15]. For other interesting attempts to describe a presentation of $H^*(S_{\mathcal{C}}, \mathbb{C})$ via differential forms see [21].

Fix a resolution $\pi : \bar{S} \rightarrow \mathbb{P}^2$ of the singularities of \mathcal{C} in \mathbb{P}^2 such that $\bar{\mathcal{C}}$, the reduced $\pi^*\mathcal{C}$, is a simple normal crossing divisor in \bar{S} . Denote by $\bar{\mathcal{C}}^{[k]}$, $k = 0, 1, 2$ the disjoint union of codimension k intersections of components of $\bar{\mathcal{C}}$. Let ${}^\ell R_m^{[k]}$ be the residue operator on $W_\ell^{[m]}$, the weight filtration in the sheaf of logarithmic forms on $\bar{\mathcal{C}}^{[m]}$ with respect to divisor $\bar{\mathcal{C}}^{[m+1]}$. A crucial fact is that these filtrations are compatible with the exterior differential d and that the quotient residues ${}^\ell \tilde{R}_m^{[k]}$ defined on $W_\ell^{[m]}/W_{\ell-1}^{[m]}$ induce isomorphisms in d -cohomology. In particular, for $k = 1, 2$ one has the residue operators $R^{[k]} = {}^k R_0^{[k]}$ taking the weight filtration $W_k = W_k^{[0]}$ on the logarithmic forms on \bar{S} with respect to the divisor $\bar{\mathcal{C}}$ (denoted by $\mathcal{A}_{\bar{S}}(\log \langle \bar{\mathcal{C}} \rangle)$) into the differential forms $\mathcal{A}_{\bar{\mathcal{C}}^{[k]}} = W_0^{[k]}$. Now passing to d -cohomology in the exact sequence of complexes $0 \rightarrow W_{i-1} \rightarrow W_i \rightarrow W_i/W_{i-1} \rightarrow 0$, and using the resolution π and deRham isomorphisms, we can construct from the coboundary maps the following residue maps:

$$\text{Res}^{[i]} : H^i(S_{\mathcal{C}}, \mathbb{C}) \rightarrow H^0(\bar{\mathcal{C}}^{[i]}), i = 1, 2.$$

These allow us to capture the cohomology groups we are interested in $H^i(S_{\mathcal{C}}, \mathbb{C})$, $i = 1, 2$.

First of all, $\text{Res}^{[1]} : H^1(S_{\mathcal{C}}, \mathbb{C}) \rightarrow H^0(\bar{\mathcal{C}}^{[1]})$ turns out to be an injection. Then a basis for $H^1(S_{\mathcal{C}}, \mathbb{C})$ can be chosen to be the cohomology classes of the logarithmic 1-forms $\sigma_i = d(\log \frac{C_i}{C_0^{d_i}})$, $1 \leq i \leq r$, where C_i are the irreducible components of \mathcal{C} and C_0 is a transversal line at infinity. This condition is not strictly necessary, but we use it for technical reasons. For a general description see Remark 3.32.

The map $\text{Res}^{[2]} : H^2(S_{\mathcal{C}}, \mathbb{C}) \rightarrow H^0(\bar{\mathcal{C}}^{[2]})$ will not be an injection in general, unless all the components of \mathcal{C} are rational. Nevertheless we can find a decomposition of $H^2(S_{\mathcal{C}}, \mathbb{C})$ of the form $E_{\mathcal{C}} \oplus K_{\mathcal{C}} \oplus \overline{K}_{\mathcal{C}}$, where $\ker \text{Res}^{[2]} = K_{\mathcal{C}} \oplus \overline{K}_{\mathcal{C}}$, with $K_{\mathcal{C}}$ a g -dimensional vector space of classes of holomorphic 2-forms of weight 1 such that ${}^1 \tilde{R}_0^{[1]} K_{\mathcal{C}}$ exhausts the holomorphic 1-forms on $\bar{\mathcal{C}}^{[1]}$, and $\overline{K}_{\mathcal{C}}$ is the conjugate of $K_{\mathcal{C}}$. Note that $\overline{K}_{\mathcal{C}}$ will necessarily consist of classes having non-holomorphic representatives. Now $E_{\mathcal{C}}$ is a vector space generated by the classes of certain log-resolution logarithmic 2-forms which are constructed by the same method employed in [5] for the rational arrangements case. The basic ingredients are logarithmic ideals associated with the resolution trees appearing in the construction of π , and ideal sheaves associated to pairs of branches at the singular points of \mathcal{C} . The choice of the log-resolution logarithmic 2-forms is done by imposing appropriate $\text{Res}^{[2]}$ normalizing conditions.

The important feature of the decomposition $H^2(S_{\mathcal{C}}, \mathbb{C}) = E_{\mathcal{C}} \oplus K_{\mathcal{C}} \oplus \overline{K}_{\mathcal{C}}$ is that the cup products $H^1(S_{\mathcal{C}}, \mathbb{C}) \cup H^1(S_{\mathcal{C}}, \mathbb{C})$ of 1-classes land in $E_{\mathcal{C}}$. Moreover, by a residue computation we determine the map $H^1(S_{\mathcal{C}}, \mathbb{C}) \cup H^1(S_{\mathcal{C}}, \mathbb{C}) \rightarrow E_{\mathcal{C}}$ on the constructed generators and see that it depends only on the degrees of the components of \mathcal{C} and the intersection numbers of the local branches at the singular points of \mathcal{C} . By another residue computation we determine the relations among the generators of $E_{\mathcal{C}}$. Finally, adding the trivial relations $H^1(S_{\mathcal{C}}, \mathbb{C}) \cup H^2(S_{\mathcal{C}}, \mathbb{C}) = 0$ we obtain a presentation for the cohomology algebra $H^*(S_{\mathcal{C}}, \mathbb{C})$.

If in choosing the 2-forms in $E_{\mathcal{C}}$ we also impose appropriate $\text{Res}^{[1]}$ normalizing conditions, then the relations in $H^1(S_{\mathcal{C}}, \mathbb{C}) \cup H^1(S_{\mathcal{C}}, \mathbb{C}) = E_{\mathcal{C}}$ will be satisfied at the differential forms level. We thus obtain an analogue of the Brieskorn lemma in the theory of hyperplane arrangements, thereby obtaining an embedding of the algebra of differential forms on $S_{\mathcal{C}}$ into the cohomology algebra $H^*(S_{\mathcal{C}}, \mathbb{C})$. This immediately implies the formality of $S_{\mathcal{C}}$.

Once the cohomology algebra $H^*(S_{\mathcal{C}}, \mathbb{C})$ is determined we explore a tower of invariants associated to it, known as the resonance varieties of $S_{\mathcal{C}}$. These are the cohomology jumping loci $\mathcal{R}_i(\mathcal{C})$ of the multiplication in $H^*(S_{\mathcal{C}}, \mathbb{C})$ by a class of degree 1, and are subvarieties of the affine space $H^1(S_{\mathcal{C}}, \mathbb{C})$. As a consequence of our main result we obtain that the resonance varieties $\mathcal{R}_i(\mathcal{C})$ depend only on the weak combinatorial type $W_{\mathcal{C}}$ of the curve \mathcal{C} .

The resonance varieties $\mathcal{R}_i(\mathcal{C})$ are strongly related to the twisted cohomology jumping loci $\Sigma_i(\mathcal{C})$ of \mathcal{C} , which stratify the torus of the rank 1 local systems on $S_{\mathcal{C}}$ by the dimension of the first twisted cohomology of $S_{\mathcal{C}}$. In fact, it follows from [9], that $\mathcal{R}_i(\mathcal{C})$ is isomorphic to the tangent cone to $\Sigma_i(\mathcal{C})$ at the trivial character 1. In order to generalize the description from [12, 23] of the irreducible components of $\Sigma_i(\mathcal{C})$ passing through 1 we generalize their notion of combinatorial pencil from line arrangements to general curves.

2. SETTINGS

2.1. C^∞ log complex of quasi-projective algebraic varieties. The aim of this section is to present an appropriate setting for the study of the cohomology ring of the complement to plane algebraic curves. This includes definitions and basic properties of logarithmic sheaves and the definition of a very useful operator on these sheaves: the Poincaré residue operator. Most definitions and results in this section follow from [15, Chapter 5].

Let X be a smooth, quasi-projective algebraic variety of dimension n over \mathbb{C} and \overline{X} be a smooth compactification of X . We will assume \overline{X} to be a smooth projective variety such that $X = \overline{X} \setminus \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ is a simple normal crossing divisor, that is, a union $\overline{\mathcal{D}}_1 \cup \dots \cup \overline{\mathcal{D}}_N$ of smooth divisors on \overline{X} with normal crossings. The condition of normal crossing on $\overline{\mathcal{D}}$ means that locally at $P \in \overline{X}$, the divisor $\overline{\mathcal{D}}$ is given by

$$\{(z_1, \dots, z_n) \mid z_{i_1} \cdots z_{i_m} = 0\} = \{(z_1, \dots, z_n) \mid z_{I_P} = 0\},$$

where $I_P = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. Each coordinate of I_P must correspond locally to a unique global component of $\overline{\mathcal{D}}$ (since each component $\overline{\mathcal{D}}_i$ is smooth). We will use a tilde as in \tilde{I}_P to indicate the ordered set of such subindices, that is, $\tilde{I}_P \subset \{1, \dots, N\}$.

Definition 2.1. Let $\mathcal{A}_{\overline{X}}$ be the sheaf of C^∞ forms on \overline{X} . Denote by $\mathcal{A}_{\overline{X}}^0$ the sheaf of C^∞ functions on \overline{X} . Note that $\mathcal{A}_{\overline{X}}$ is a sheaf of graded algebras over $\mathcal{A}_{\overline{X}}^0$. The *sheaf of C^∞ log forms* $\mathcal{A}_{\overline{X}}(\log \langle \overline{\mathcal{D}} \rangle)$ can be defined locally at a point $P \in \overline{X}$ as the graded algebra over $(\mathcal{A}_{\overline{X}}^0)_P$ of C^∞ forms $\varphi \in (\mathcal{A}_{\overline{X}})_P$ such that

$$z_{I_P} \varphi \quad \text{and} \quad z_{I_P} d\varphi$$

are C^∞ forms in $(\mathcal{A}_{\overline{X}})_P$.

A form φ on $U \subset \overline{X}$ shall be called *logarithmic on U (with respect to $\overline{\mathcal{D}}$)* if $\varphi \in \mathcal{A}_{\overline{X}}(\log \langle \overline{\mathcal{D}} \rangle)(U)$.

The sheaf $\mathcal{A}_{\overline{X}}(\log \langle \overline{\mathcal{D}} \rangle)$ is a locally free and finitely generated $\mathcal{A}_{\overline{X}}^0$ -algebra, as follows from

Lemma 2.2 ([15, Lemma 5.7]). $\mathcal{A}_{\overline{X}}(\log \langle \overline{\mathcal{D}} \rangle)(U_P) \cong \mathcal{A}_{\overline{X}}(U_P) \left\{ \frac{dz_i}{z_i} \right\}_{i \in I_P}$

By definition, $\mathcal{A}_{\overline{X}}(\log(\overline{\mathcal{D}}))$ is closed under the exterior derivation d . This lemma shows that it is in fact closed under the exterior product and generated by $\mathcal{A}_{\overline{X}}^1(\log(\overline{\mathcal{D}}))$.

In what follows, a weight filtration is defined in this sheaf of graded algebras that is compatible with the differential d .

Definition 2.3. If $\ell \geq 0$ we shall define the *sheaf of C^∞ log forms of weight ℓ* as the subsheaf of $\mathcal{A}_{\overline{X}}(\log(\overline{\mathcal{D}}))$ given locally as the $(\mathcal{A}_{\overline{X}})_P^0$ -submodule of $\mathcal{A}_{\overline{X}}(\log(\overline{\mathcal{D}}))_P$ of those forms φ such that

$$\varphi \in \sum_{\substack{I \subset I_P, \\ |I| \leq \ell}} \mathcal{A}_{\overline{X}} \left\{ \frac{dz_i}{z_i} \right\}_{i \in I}.$$

Such a sheaf will be denoted by $\mathcal{W}_\ell := \mathcal{W}_\ell(\mathcal{A}_{\overline{X}}(\log(\overline{\mathcal{D}})))$. Otherwise, that is, if $\ell < 0$, we will assume $\mathcal{W}_\ell := \{0\}$.

Remark 2.4. Note that $\mathcal{W}_\ell \subset \mathcal{W}_{\ell+1}$, $d\mathcal{W}_\ell \subset \mathcal{W}_\ell$, and $\mathcal{W}_\ell \wedge \mathcal{W}_{\ell'} \subset \mathcal{W}_{\ell+\ell'}$ are obvious consequences of Definition 2.3.

Notations 2.5.

- (1) Let us denote by $\overline{\mathcal{D}}^{[k]}$ the disjoint union of codimension k intersections of components of $\overline{\mathcal{D}}$, that is,

$$\overline{\mathcal{D}}^{[k]} := \bigsqcup_{|I|=k} \overline{\mathcal{D}}_I$$

where $\overline{\mathcal{D}}_I = \cap_{i \in I} \overline{\mathcal{D}}_i$.

- (2) There is an injection $\overline{\mathcal{D}}_I \xrightarrow{i_I} \overline{X}$ for each $\overline{\mathcal{D}}_I \in \overline{\mathcal{D}}^{[k]}$. Denoting by i_k the corresponding map on $\overline{\mathcal{D}}^{[k]}$, one has the following sheaf on \overline{X}

$$\mathcal{A}_{\overline{\mathcal{D}}^{[k]}}^* = (i_k)_* \bigoplus_{|I|=k} \mathcal{A}_{\overline{\mathcal{D}}_I}^*$$

Definition 2.6. Under the notations above, the *Poincaré residue operator*

$$R^{[k]} : \mathcal{W}_k(\mathcal{A}_{\overline{X}}^*(\log(\overline{\mathcal{D}}))) \longrightarrow \mathcal{A}_{\overline{\mathcal{D}}^{[k]}}^{*-k}$$

can be defined locally by

$$R^{[k]} \left(\alpha_P \wedge \frac{dz_I}{z_I} \right) = (-1)^{\sigma(\tilde{I})} \alpha_P|_{\overline{\mathcal{D}}_I},$$

where:

- i) $\frac{dz_I}{z_I}$ denotes $\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}$, and
- ii) If $\tilde{I} = \{\tilde{i}_1, \dots, \tilde{i}_k\} \subset \{1, \dots, N\}$, then $\sigma(\tilde{I}) := \text{sign}(\tilde{i}_{k+1}, \dots, \tilde{i}_N, \tilde{i}_1, \dots, \tilde{i}_k)$, where $\tilde{i}_{k+1} < \cdots < \tilde{i}_N$ are the ordered elements of $\{1, \dots, N\} \setminus \tilde{I}$.

Remark 2.7. Note that for any $\overline{\mathcal{D}}_{I'}$ and $\overline{\mathcal{D}}_I$ with $|I'| = k+1$ and $|I| = k$ one can define a smooth divisor on $\overline{\mathcal{D}}_I$ as follows:

$$\overline{\mathcal{D}}_I|_{\overline{\mathcal{D}}_{I'}} := \begin{cases} \overline{\mathcal{D}}_{I'} & \text{if } I \subset I' \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, the union

$$\overline{\mathcal{D}}_I|_{\overline{\mathcal{D}}^{[k+1]}} := \sum_{|I'|=k+1} \overline{\mathcal{D}}_I|_{\overline{\mathcal{D}}_{I'}}$$

provides a simple normal crossing divisor in $\overline{\mathcal{D}}_I \subset \overline{\mathcal{D}}^{[k]}$. Hence $\overline{\mathcal{D}}^{[k]}$ can be regarded as a disjoint union of smooth compact algebraic varieties, each component containing a divisor with normal crossings. Therefore one can consider the sheaf of C^∞ log forms on each smooth algebraic variety $\overline{\mathcal{D}}_I$ with respect to $\overline{\mathcal{D}}_I|_{\overline{\mathcal{D}}^{[k+1]}}$, denoted by $\mathcal{A}_{\overline{\mathcal{D}}_I}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle)$.

Definition 2.8. By means of the inclusions $\overline{\mathcal{D}}_I \xrightarrow{i_k} \overline{\mathcal{X}}$ one can also define *log sheaves on $\overline{\mathcal{D}}^{[k]}$ relative to $\overline{\mathcal{D}}^{[k+1]}$* as subsheaves of the direct sum of log sheaves for each component satisfying certain compatibility relations. That is,

$$\mathcal{A}_{\overline{\mathcal{D}}^{[k]}}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle) \subset \bigoplus_{|I|=k} \mathcal{A}_{\overline{\mathcal{D}}_I}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle),$$

defined by the following local condition. For any strings I_1, I_2, I'_1, I'_2 such that $|I_i| = |I'_i| = k_i$, $i = 1, 2$ and $\{I_1 + I_2\} = \{I'_1 + I'_2\}$, and for any pair of forms

$$\alpha_P \frac{dz_{I_2}}{z_{I_2}} \in \left(\mathcal{A}_{\overline{\mathcal{D}}_{I_1}}^*(\log\langle\overline{\mathcal{D}}^{[k_1+1]}\rangle) \right)_P, \quad \text{and} \quad \beta_P \frac{dz_{I'_2}}{z_{I'_2}} \in \left(\mathcal{A}_{\overline{\mathcal{D}}_{I'_1}}^*(\log\langle\overline{\mathcal{D}}^{[k_1+1]}\rangle) \right)_P,$$

one has

$$(1) \quad (-1)^{\sigma(\tilde{I}_1)} (-1)^{\sigma(\widetilde{I_2+I_1})} \alpha_P|_{\overline{\mathcal{D}}_{I_1}} = (-1)^{\sigma(\tilde{I}'_1)} (-1)^{\sigma(\widetilde{I'_2+I'_1})} \beta_P|_{\overline{\mathcal{D}}_{I'_1}},$$

where \tilde{I}_i and \tilde{I}'_i are defined as in 2.6 and $I + I'$ denotes juxtaposition of strings. For simplification this sheaf will be denoted by $\mathcal{A}_k^*(\log\langle\overline{\mathcal{D}}\rangle)$ and its projection on $\overline{\mathcal{D}}_I$ (for $|I| = k$) by $\mathcal{A}_{k,I}^*(\log\langle\overline{\mathcal{D}}\rangle)$.

There also exists an obvious weight filtration on $\mathcal{A}_k(\log\langle\overline{\mathcal{D}}\rangle)$, denoted by $\mathcal{W}_\ell^{[k]}$. Note that $\mathcal{W}_\ell^{[0]} = \mathcal{W}_\ell(\mathcal{A}_{\overline{\mathcal{X}}}(\log\langle\overline{\mathcal{D}}\rangle))$ and $\mathcal{W}_0^{[k]} = \mathcal{A}_{\overline{\mathcal{D}}^{[k]}}$. The compatibility relations described in (1) allow for a generalization of the Poincaré residue operator to all the log sheaves relative to $\overline{\mathcal{D}}$, namely

$$(2) \quad {}^\ell R_m^{[k]} : \mathcal{W}_\ell^{[m]} \longrightarrow \mathcal{W}_{\ell-k}^{[m+k]}.$$

In order to give a local description of ${}^\ell R_m^{[k]}$ let us consider a point $P \in \overline{\mathcal{X}}$ and a form $\varphi \in (\mathcal{A}_k^*(\log\langle\overline{\mathcal{D}}\rangle))_P$. Let us denote by $\left({}^\ell R_m^{[k]} \varphi\right)_I$ the coordinate of ${}^\ell R_m^{[k]} \varphi$ on $\mathcal{A}_{k+m,I}^{*-k}(\log\langle\overline{\mathcal{D}}\rangle)_P$, where $|I| = m + k$. In order to calculate this coordinate take two disjoint strings I_1 and I_2 such that $|I_1| = m$, and $\overline{\mathcal{D}}_I = \{z_{I_1} z_{I_2} = 0\}$ – and hence $|I_2| = k$. The form φ can be written as

$$\alpha \frac{dz_{I_2}}{z_{I_2}} \in (\mathcal{A}_{k,I_1}^*(\log\langle\overline{\mathcal{D}}\rangle))_P.$$

Thus one can define

$$\left({}^\ell R_m^{[k]} \varphi\right)_I := (-1)^{\sigma(\tilde{I}_1)} (-1)^{\sigma(\widetilde{I_2+I_1})} \alpha|_{\overline{\mathcal{D}}_I}.$$

Again by (1) the definition of $\left({}^\ell R_m^{[k]} \varphi\right)_I$ does not depend on the choice of I_1 and I_2 .

The main result about these generalized residue maps, which will be intensively used throughout the paper, is the following

Theorem 2.9 ([15, Theorem 5.15] and [5, Theorem 1.28]). *Any generalized residue mapping*

$${}^\ell \tilde{R}_m^{[k]} : (W_\ell^{[m],*} / W_{\ell-1}^{[m],*}) \longrightarrow (W_{\ell-k}^{[m+k],*-k} / W_{\ell-k-1}^{[m+k],*-k})$$

on the complex of global sections induces an isomorphism on d -cohomology. Moreover,

$${}^{\ell-k_1} \tilde{R}_{m+k_1}^{[k_2]} \circ {}^\ell \tilde{R}_m^{[k_1]} = {}^\ell \tilde{R}_m^{[k_1+k_2]}$$

2.2. The spaces $H^k(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{C})$ and the residue maps. As a general setting, let S be a smooth compact surface, $\mathcal{C} \subset S$ a reduced divisor. Let us denote by $S_{\mathcal{C}}$ the complement of \mathcal{C} in S . Consider a resolution $\pi : \bar{S}_{\mathcal{C}} \rightarrow S$ of \mathcal{C} in S such that $\bar{S}_{\mathcal{C}}$ is a compactification of $S_{\mathcal{C}}$ by a simple normal crossing divisor, and let $\bar{\mathcal{C}}$ be the reduced structure of $\pi^*\mathcal{C}$. Note that $S_{\mathcal{C}}$ is isomorphic to $\bar{S}_{\mathcal{C}} \setminus \bar{\mathcal{C}}$ via π .

Definition 2.10. A *log-resolution logarithmic form* on \mathcal{C} at $P \in S$ is a differential form $\varphi \in (\mathcal{A}_{S_{\mathcal{C}}}^*)_P$ such that $\pi^*(\varphi)_P \in \mathcal{A}^*(\log(\bar{\mathcal{C}}))_P$, that is, $\pi_*\mathcal{A}_{S_{\mathcal{C}}}^*$, which will be denoted by $\mathcal{A}^{\log}(\mathcal{C})$.

Remark 2.11. Consider $\mathcal{C} \subset S$ a simple normal crossing divisor, $P \in S$, and $\varphi \in \mathcal{A}^*(\log(\mathcal{C}))_P$ a differential logarithmic form. Denote by $\pi : \bar{S} \rightarrow S$ the blow-up of P in S . Note that $\pi^*\varphi$ is also a logarithmic form on $\pi^{-1}\mathcal{C}$ at any point $Q \in \pi^{-1}(P)$. This, together with the fact that any two sequences of blow-ups of S are dominated by a third one, implies that the notion of log-resolution logarithmic form on \mathcal{C} is independent of the given embedded resolution of \mathcal{C} .

Note that $\mathcal{A}_S^{\log}(\mathcal{C}) \subset \mathcal{A}_S(\mathcal{C})$, where $\mathcal{A}_S(\mathcal{C})$ is the classical sheaf of logarithmic differential forms on \mathcal{C} locally defined as

$$(\mathcal{A}_S(\mathcal{C}))_P := \{\varphi \in (\mathcal{A}_{S_{\mathcal{C}}}^*)_P \mid C_P\varphi \in (\mathcal{A}_{\bar{S}_{\mathcal{C}}}^*)_P, \quad C_P d\varphi \in (\mathcal{A}_{\bar{S}_{\mathcal{C}}}^*)_P\},$$

where C_P is a reduced equation of \mathcal{C} at P .

In fact $\mathcal{A}_S^{\log}(\mathcal{C})$ is the biggest subsheaf of $\mathcal{A}_S(\mathcal{C})$ that is stable under blow-ups. Moreover, by Lemma 2.2, $\mathcal{A}_S^{\log}(\mathcal{C})(\bar{S}_{\mathcal{C}})$ is always a free module.

Construction 2.12. Let $\mathcal{C} \subset \mathbb{P}^2$ be an algebraic curve with irreducible components $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_r$ and let us fix $\pi : \bar{S}_{\mathcal{C}} \rightarrow \mathbb{P}^2$ a resolution of the singularities of \mathcal{C} so that the reduced divisor $\bar{\mathcal{C}} = (\pi^*(\mathcal{C}))_{\text{red}}$ is a simple normal crossing divisor on $\bar{S}_{\mathcal{C}}$ as described in section 2.1.

Consider the following short exact sequence of complexes $0 \rightarrow W_{i-1} \rightarrow W_i \rightarrow W_i/W_{i-1} \rightarrow 0$ where W_i denotes the complex $(W_i^{[0],*} \mathcal{A}_{\bar{S}_{\mathcal{C}}}(\log(\bar{\mathcal{C}})), d)$. Let us consider its corresponding long exact sequence of d -cohomology

$$(3) \quad \dots \rightarrow H^{k-1}(W_i/W_{i-1}) \rightarrow H^k(W_{i-1}) \rightarrow H^k(W_i) \xrightarrow{\delta^k} H^k(W_i/W_{i-1}) \rightarrow H^{k+1}(W_{i-1}) \rightarrow \dots$$

By the de Rham Theorem and Theorem 2.9, one can define the *residue map* $\text{Res}^{[i]}$ as the composition of the following:

$$(4) \quad H^i(\mathbb{P}^2; \pi_* W_i) \xrightarrow{DR} H^i(S_{\mathcal{C}}; \mathbb{C}) \cong H^i(\bar{S}_{\mathcal{C}}; \mathbb{C}) \xrightarrow{DR} H^i(\bar{S}_{\mathcal{C}}, W_i) \xrightarrow{\delta^i} H^i(\bar{S}_{\mathcal{C}}, W_i/W_{i-1}) \xrightarrow{R[i]} H^0(\bar{\mathcal{C}}^{[i]}; \mathbb{C})$$

Proposition 2.13 ([5, Proposition 2.2]). *Let \mathcal{C} be an algebraic plane curve, then*

$$H^2(S_{\mathcal{C}}; \mathbb{C}) \cong H_1(\mathcal{C}; \mathbb{C}), \text{ and } H^1(S_{\mathcal{C}}; \mathbb{C}) \cong H_2(\mathcal{C}; \mathbb{C})/\mathbb{C}.$$

Notation 2.14. Let Y be a topological space. In what follows we will denote by $h_i(Y)$ (resp. $h^i(Y)$) the dimension of the vector space $H_i(Y; \mathbb{C})$ (resp. $H^i(Y; \mathbb{C})$). Note that, by the Universal Coefficient Theorem, $h_i(Y) = h^i(Y)$.

One has the following result.

Proposition 2.15. *The first residue map $H^1(S_{\mathcal{C}}) \xrightarrow{\text{Res}^{[1]}} H^0(\bar{\mathcal{C}}^{[1]})$ as defined in Construction 2.12 is injective. On the other hand*

$$\ker \left(H^2(S_{\mathcal{C}}) \xrightarrow{\text{Res}^{[2]}} H^0(\bar{\mathcal{C}}^{[2]}) \right) = H^2(\mathbb{P}^2; \pi_* W_1) \cong \mathbb{C}^{2g},$$

where $g = \sum_{i=1}^r g(\mathcal{C}_i)$ is the sum of the genera of the irreducible components of \mathcal{C} .

Proof. The injectivity of $\text{Res}^{[1]}$ follows immediately from (3) for the case $i = k = 1$, and $H^1(W_0, d) = 0$. Let us consider now (3) for $k = 2$, $i = 1$.

$$(5) \quad \begin{array}{ccccccc} H^1(W_0) & \rightarrow & H^1(W_1) & \rightarrow & H^1(W_1/W_0) & \rightarrow & H^2(W_0) \rightarrow H^2(W_1) \rightarrow H^2(W_1/W_0) \rightarrow H^3(W_0) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ H^1(\overline{S}_C) & & H^1(S_C) & & H^0(\overline{\mathcal{C}}^{[1]}) & & H^2(\overline{S}_C) & & H^1(\overline{\mathcal{C}}^{[1]}) \rightarrow H^3(\overline{S}_C) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathbb{C}^r & & \mathbb{C}^{r+e+1} & & \mathbb{C}^{e+1} & & \mathbb{C}^{2g} \rightarrow 0 \end{array}$$

where e is the number of exceptional components in the resolution of \mathcal{C} . The equalities on the second column are a consequence of de Rham and Proposition 2.13. The others are a consequence of $H^k(W_0) = H^k(\overline{S}_C)$, de Rham, and Theorem 2.9.

Computing the Euler characteristic of this long exact sequence one obtains that $H^2(W_1, d) \cong \mathbb{C}^{2g}$ and therefore, using (3) for the case $i = k = 2$ one obtains

$$0 \rightarrow H^2(W_1) = \mathbb{C}^{2g} \rightarrow H^2(W_2) = H^2(S_C) \xrightarrow{\text{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]}) \rightarrow H^3(W_1) \rightarrow \dots$$

which proves that $\ker \left(H^2(S_C) \xrightarrow{\text{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]}) \right) = H^2(\overline{S}_C; W_1) \cong \mathbb{C}^{2g}$. Finally, by the Leray spectral sequence, since all these sheaves are flasque, one has the projection formula $H^2(\overline{S}_C; W_1) \cong H^2(\mathbb{P}^2; \pi_* W_1)$. \square

2.3. Classical combinatorics. In this paragraph we just want to give a general outline on the classical concept of *combinatorial type* of a curve. This concept is generally accepted and used, but is seldom explicitly defined. In [2] there is a detailed explanation on the matter. For the sake of completeness, we summarize the main ideas.

Definition 2.16. Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane projective curve. The *combinatorial type* of \mathcal{C} is given by a sextuplet

$$K_{\mathcal{C}} := (\mathbf{r}_{\mathcal{C}}, \bar{d}_{\mathcal{C}}, \text{Sing}(\mathcal{C}), \Sigma_{\mathcal{C}}^{\text{top}}, \sigma_{\mathcal{C}}^{\text{top}}, \{\Delta_{\mathcal{C},P}, \phi_{\mathcal{C},P}\}_{P \in \text{Sing}(\mathcal{C})}),$$

where:

- (i) $\mathbf{r}_{\mathcal{C}}$ is the set of irreducible components of \mathcal{C} ,
- (ii) $\bar{d}_{\mathcal{C}} : \mathbf{r}_{\mathcal{C}} \rightarrow \mathbb{N}$ is the list of degrees,
- (iii) $\text{Sing}(\mathcal{C})$ is the set of singular points of \mathcal{C} ,
- (iv) $\Sigma_{\mathcal{C}}^{\text{top}}$ is the set of topological types of $\text{Sing}(\mathcal{C})$,
- (v) $\sigma_{\mathcal{C}}^{\text{top}} : \text{Sing}(\mathcal{C}) \rightarrow \Sigma_{\mathcal{C}}^{\text{top}}$ assigns to each singular point its topological type,
- (vi) $\Delta_{\mathcal{C},P}$ is the set of local branches of \mathcal{C} at $P \in \text{Sing}(\mathcal{C})$, (a local branch can be seen as an arrow in the dual graph of the minimal resolution of \mathcal{C} at P , see [11, Chapter II.8] for details) and $\phi_{\mathcal{C},P} : \Delta_{\mathcal{C},P} \rightarrow \mathbf{r}_{\mathcal{C}}$ assigns to each local branch the global irreducible component that contains it.

We say that two curves \mathcal{C}_1 and \mathcal{C}_2 have the *same combinatorial type* (or simply the *same combinatorics*) if their combinatorial data $K_{\mathcal{C}_1}$ and $K_{\mathcal{C}_2}$ are equivalent, that is, if $\Sigma_{\mathcal{C}_1}^{\text{top}} = \Sigma_{\mathcal{C}_2}^{\text{top}}$, and there exist bijections:

- (1) $\varphi_{\mathbf{r}} : \mathbf{r}_{\mathcal{C}_1} \rightarrow \mathbf{r}_{\mathcal{C}_2}$,
- (2) $\varphi_{\text{Sing}} : \text{Sing}(\mathcal{C}_1) \rightarrow \text{Sing}(\mathcal{C}_2)$, and
- (3) $\varphi_P : \Delta_{\mathcal{C}_1,P} \rightarrow \Delta_{\mathcal{C}_2,\varphi_{\text{Sing}}(P)}$ (the restriction of a bijection of dual graphs) for each $P \in \text{Sing}(\mathcal{C}_1)$

such that:

- (1) $\bar{d}_{\mathcal{C}_1} = \bar{d}_{\mathcal{C}_2} \circ \varphi_{\mathbf{r}}$,
- (2) $\sigma_{\mathcal{C}_1}^{\text{top}} = \sigma_{\mathcal{C}_2}^{\text{top}} \circ \varphi_{\text{Sing}}$, and
- (3) $\varphi_{\mathbf{r}} \circ \phi_{\mathcal{C}_1, P} = \phi_{\mathcal{C}_2, \varphi_{\text{Sing}}(P)} \circ \varphi_P$.

In the irreducible case, two curves have the same combinatorial type if they have the same degree and the same topological types for local singularities. On the other extreme, for line arrangements, combinatorial type is just the set of incidence relations. In higher dimensions, the concept of combinatorics still makes sense but it becomes much harder to describe, except for the case of hyperplane (or in general linear) arrangements where the incidence relations are enough to determine the combinatorial type.

The main interest and motivation for combinatorial types of curves is due to the following.

Proposition 2.17. *Consider two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$, and $T(\mathcal{C}_1), T(\mathcal{C}_2)$ their regular neighborhoods with boundary. Then the pairs $(T(\mathcal{C}_1), \mathcal{C}_1)$ and $(T(\mathcal{C}_2), \mathcal{C}_2)$ are homeomorphic if and only if \mathcal{C}_1 and \mathcal{C}_2 have the same combinatorial type.*

Proof. In one direction, the self intersections of the components of \mathcal{C}_i and the topological types of the singularities of \mathcal{C}_i are well defined and preserved under homeomorphisms of pairs $(T(\mathcal{C}_i), \mathcal{C}_i)$. This determines degrees, topological types of singularities as well as the incidence of local branches. Therefore their combinatorial types coincide. Conversely, the combinatorial type allows one to obtain the minimal resolution of the singularities, which should be homeomorphic. Since the self intersections coincide one can extend this to a homeomorphism of the tubular neighborhoods of each component (including exceptional components) and glue them along the intersections as prescribed by the multiplicities of the components. By contracting the exceptional components one can define a homeomorphism of pairs between $(T(\mathcal{C}_1), \mathcal{C}_1)$ and $(T(\mathcal{C}_2), \mathcal{C}_2)$. \square

A pair of plane curves $(\mathcal{C}_1, \mathcal{C}_2)$ such that $(T(\mathcal{C}_1), \mathcal{C}_1)$ and $(T(\mathcal{C}_2), \mathcal{C}_2)$ are homeomorphic, but $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are not (that is, whose embeddings in \mathbb{P}^2 are not homeomorphic) is called a *Zariski pair*. The existence of Zariski pairs and the search for invariants of the embedding of a curve that can tell two combinatorially-equivalent curves apart has been a very productive field of research started by O.Zariski in [27, 28] (see [2] and references therein for an extended survey on Zariski pairs).

Alternatively, one can also define a weaker concept of combinatorics as follows.

Definition 2.18. Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane projective curve. The *weak combinatorial type* of \mathcal{C} is given by a quintuplet

$$W_{\mathcal{C}} := (\mathbf{r}_{\mathcal{C}}, \text{Sing}(\mathcal{C}), \{\Delta_{\mathcal{C}, P}, \phi_{\mathcal{C}, P}, (\bullet, \bullet)_{\mathcal{C}, P}\}_{P \in \text{Sing}(\mathcal{C})}, \bar{d}_{\mathcal{C}}, \bar{g}_{\mathcal{C}}),$$

where $\mathbf{r}_{\mathcal{C}} := \{1, \dots, r\}$, $\bar{d}_{\mathcal{C}}$, $\text{Sing}(\mathcal{C})$, and $\{\Delta_{\mathcal{C}, P}, \phi_{\mathcal{C}, P}\}_{P \in \text{Sing}(\mathcal{C})}$ are defined as before, $\bar{g}_{\mathcal{C}} : \mathbf{r}_{\mathcal{C}} \rightarrow \mathbb{N}$ is the list of genera, and $(\bullet, \bullet)_{\mathcal{C}, P} : \text{SP}_{\phi_{\mathcal{C}, P}}^2(\Delta_{\mathcal{C}, P}) \rightarrow \mathbb{N}$, where $\phi_{\mathcal{C}, P}(\delta)$ is the global irreducible component containing δ , $\text{SP}_{\phi_{\mathcal{C}, P}}^2(\Delta_{\mathcal{C}, P}) := \frac{\Delta_{\mathcal{C}, P} \times \Delta_{\mathcal{C}, P}}{\Sigma_2} \setminus \Delta_{\phi_{\mathcal{C}, P}}$ is the symmetric product of $\Delta_{\mathcal{C}, P}$ outside the $\phi_{\mathcal{C}, P}$ -diagonal ($\Delta_{\phi_{\mathcal{C}, P}} := \{(\delta_1, \delta_2) \mid \phi_{\mathcal{C}, P}(\delta_1) = \phi_{\mathcal{C}, P}(\delta_2)\}$), and $(\delta_1, \delta_2)_{\mathcal{C}, P}$ represents the intersection number of δ_1 and δ_2 at P .

We say that two curves \mathcal{C}_1 and \mathcal{C}_2 have the *same weak combinatorial data* (or simply the *same combinatorics*) if their weak combinatorial types $W_{\mathcal{C}_1}$ and $W_{\mathcal{C}_2}$ are equivalent, that is, if there exist bijections:

- (1) $\varphi_{\mathbf{r}} : \mathbf{r}_{\mathcal{C}_1} \rightarrow \mathbf{r}_{\mathcal{C}_2}$,
- (2) $\varphi_{\text{Sing}} : \text{Sing}(\mathcal{C}_1) \rightarrow \text{Sing}(\mathcal{C}_2)$, and
- (3) $\varphi_P : \Delta_{\mathcal{C}_1, P} \rightarrow \Delta_{\mathcal{C}_2, \varphi_{\text{Sing}}(P)}$ (restriction of a bijection of dual graphs) for each $P \in \text{Sing}(\mathcal{C}_1)$

such that:

- (1) $\bar{d}_{\mathcal{C}_1} = \bar{d}_{\mathcal{C}_2} \circ \varphi_{\mathbf{r}}$,
- (2) $\varphi_{\mathbf{r}} \circ \phi_{\mathcal{C}_1, P} = \phi_{\mathcal{C}_2, \varphi_{\text{Sing}}(P)} \circ \varphi_P$, and
- (3) $(\delta_1, \delta_2)_{\mathcal{C}_1, P} = (\varphi_P(\delta_1), \varphi_P(\delta_2))_{\mathcal{C}_2, \varphi_{\text{Sing}}(P)}$.

It is obvious that $K_{\mathcal{C}}$ determines $W_{\mathcal{C}}$ using the intersection multiplicity formula. The converse is also true for smooth arrangements (a curve whose irreducible components are smooth), but not true in general as Example 7.1 shows.

The question immediately arises as to what extent the combinatorial type of a curve determines well-known invariants of its embedding in \mathbb{P}^2 . We will refer to such invariants as *combinatorial*. Fundamental groups of complements of curves are known not to be combinatorial as shown by Zariski in [27]. The cohomology ring of the complement of a curve was only known to be combinatorial when the curve was a line arrangement [1, 3] or more generally, a rational arrangement [5]. The purpose of the upcoming section is to prove that the cohomology ring of the complement of a curve is a combinatorial invariant.

3. COHOMOLOGY RING STRUCTURE

In what follows we will describe generators for $H^*(S_{\mathcal{C}})$. For simplicity, we will assume \mathcal{C}_0 is a transversal line. We will consider coordinates $[X : Y : Z]$ in \mathbb{P}^2 such that $\mathcal{C}_0 := \{Z = 0\}$, and define $\omega := XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY$ the contraction of the volume form in the affine space \mathbb{A}^3 by the Euler vector field.

As in the classical cases the subspace $H^1(S_{\mathcal{C}})$ is generated by the log-resolution holomorphic logarithmic 1-forms $\sigma_i := d \log \frac{C_i}{C_0^{a_i}}$, $i = 1, \dots, r$, where C_i is an equation for the component \mathcal{C}_i .

Theorem 3.1 ([5, Theorem 2.10 and 2.11]). *The 1-forms σ_i , $i = 1, \dots, r$ defined above verify the following properties:*

- (i) $\sigma_i \in W^1 \mathcal{A}_{\mathbb{P}^2}^{\log}(\mathcal{C})$,
- (ii) $\Sigma := \{\sigma_1, \dots, \sigma_r\}$ generate $H^1(\overline{S}_{\mathcal{C}})$ as a vector space, and
- (iii) $\left(\text{Res}^{[1]} \sigma_i \right)_{\tilde{\mathcal{C}}_j} = \begin{cases} (-1)^{r-i} & \text{if } i = j \\ 0 & \text{if } i \neq j \neq 0 \\ (-1)^{r+1} d_i & \text{if } j = 0. \end{cases}$

In order to obtain generators for $H^2(S_{\mathcal{C}})$ we can proceed as in the rational case. Let us first recall the concepts of multiplicity trees and logarithmic trees.

Definitions 3.2.

- (i) Let $f \in \mathbb{C}\{x, y\}$ be a germ of a holomorphic function at P whose set of zeroes is a germ of curve $V_f \subset S_0$ with an isolated singularity at the point P . Consider the sequence of

blow-ups

$$S_0 \xleftarrow{\varepsilon_1} S_1 \xleftarrow{\varepsilon_2} S_2 \xleftarrow{\varepsilon_3} \dots \xleftarrow{\varepsilon_m} S_m = \overline{S}$$

in the resolution of S_0 at P , and denote by π_k the composition of the first k blow-ups $\pi_k = \varepsilon_k \circ \dots \circ \varepsilon_1$. The germ of curve $\tilde{V}_{f,k} = \pi_k^{-1}(V_f \setminus \{P\})$ shall be called the *strict transform of V_f in S_k* and its equation denoted by \tilde{f}_k . The divisor $\pi_k^*(V_f)$ shall be denoted by $\overline{V}_{f,k}$ and called the *total transform of V_f in S_k* . For simplicity let us write $\tilde{V}_f := \tilde{V}_{f,m}$ and $\overline{V}_f := \overline{V}_{f,m}$. The exceptional divisor in S_k resulting from the blow-up of a point in S_{k-1} shall be denoted by E_k and the points $P_k^1, \dots, P_k^{N_k}$ in $E_k \cap \tilde{V}_{f,k}$ called the *infinitely near points to P in E_k* . For convenience, the point P is also considered to be infinitely near to itself. Finally, the multiplicity of $\tilde{V}_{f,k} \subset S_\ell$ at the point P_k^ℓ shall be denoted by $\nu_{P_k^\ell}(f)$, i.e.

$$\nu_{P_k^\ell}(f) := \text{mult}_{P_k^\ell}(\tilde{V}_{f,k}).$$

(ii) To each resolution of singularities π one can assign the *multiplicity tree of π at P* –denoted by $\mathcal{T}_P(\pi, f)$, or simply by $\mathcal{T}_P(f)$ if the resolution π of S_0 is fixed. $\mathcal{T}_P(f)$ is a tree with weights at each vertex and is defined as follows.

- a. The vertices of $\mathcal{T}_P(f)$ are in bijection with the infinitely near points to P .
- b. Two vertices of $\mathcal{T}_P(f)$, say Q_1 and Q_2 , are joined by an edge if and only if one of the points, say Q_2 , belongs to S_k for some k , the other point Q_1 belongs to S_{k-1} and $Q_2 \in \varepsilon_k^{-1}(Q_1)$.
- c. For convenience, this tree is considered to simply be a vertex if P is not a singular point of f . If $f(P) \neq 0$, then $\mathcal{T}_P(f) := \emptyset$.
- d. The weight $w(\mathcal{T}_P(f), Q)$ of a vertex Q is $\nu_Q(f)$.

The *extended multiplicity tree at P* , denoted by $\tilde{\mathcal{T}}_P(f)$ contains the multiplicity tree $\mathcal{T}_P(f)$ as a subtree. It can be constructed as follows.

- a'. The vertices of $\tilde{\mathcal{T}}_P(f)$ correspond to the points of $\text{Sing}(\overline{V}_f)$. Note that each extra vertex corresponds to a non-infinitely near point in $\text{Sing}(\overline{V}_f)$, that is, an intersection of two exceptional divisors, say E_i and E_j . They shall be denoted by e_{ij} with the convention $i < j$.
- b'. A vertex e_{ij} is joined to another vertex Q if Q belongs to S_k for a certain k and $e_{ij} \in \varepsilon_k^{-1}(Q)$. Note that necessarily $k = j$. The remaining edges come from the fact that $\mathcal{T}_P(f)$ is a subtree of $\tilde{\mathcal{T}}_P(f)$.
- c'. The weight at each e_{ij} shall be defined to be 0. The weight of each of the remaining vertices coincides with that of $\mathcal{T}_P(f)$.

(iii) The set of vertices $|\tilde{\mathcal{T}}_P(f)|$ of an extended multiplicity tree $\tilde{\mathcal{T}}_P(f)$ is endowed with a partial order as follows. Consider P as the root of the tree and direct the edges of the tree towards P . In this *directed tree*, a point Q_2 is said to be *greater* than Q_1 –denoted $Q_2 \geq Q_1$ – if there is a directed path from Q_2 to Q_1 . In graph theory this situation is commonly described by calling Q_2 an *ancestor* of Q_1 , or Q_1 a *descendant* of Q_2 . Given a set of points $\{P_1, \dots, P_n\} \subset \tilde{\mathcal{T}}_P(f)$ one can define

$$\text{Asc}(P_1, \dots, P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \geq P_i \ i = 1, \dots, n\},$$

and

$$\text{Desc}(P_1, \dots, P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \leq P_i \ i = 1, \dots, n\}.$$

Multiplicity trees are *quasi-strongly connected trees*, which means that the set of common descendants $\text{Desc}(P_1, \dots, P_n)$ is non empty and inherits a linear order from $\mathcal{T}_P(f)$. The maximal element in $\text{Desc}(P_1, \dots, P_n)$ is called the *greatest common descendant* and is denoted by $\gcd(P_1, \dots, P_n)$.

(iv) The *degree* of a weighted tree \mathcal{T} shall be defined as

$$(6) \quad \deg(\mathcal{T}) := \sum_{Q \in |\mathcal{T}|} \binom{w(\mathcal{T}, Q) + 1}{2},$$

where $w(\mathcal{T}, Q)$ denotes the weight of \mathcal{T} at Q .

Note that if $\mathcal{T} = \mathcal{T}_P(f)$, then $\deg(\mathcal{T})$ is the δ -invariant of the singularity of f at P .

(v) In order to simplify, we shall write $\mathcal{T} \cong \mathcal{T}'$ for two weighted trees that are isomorphic as trees, and $\mathcal{T} = \mathcal{T}'$ (resp. $\geq, \leq, <$ or $>$) if $\mathcal{T} \cong \mathcal{T}'$ and $\hat{w}(\mathcal{T}, Q) = \hat{w}(\mathcal{T}', Q)$ (resp. $\geq, \leq, <$ or $>$) for any $Q \in |\mathcal{T}| = |\mathcal{T}'|$, where $\hat{w}(\mathcal{T}, Q) := \sum_{Q' \in \text{Desc}(Q)} w(\mathcal{T}, Q')$ (we are using the isomorphism of trees to identify the vertices). Note that $\hat{w}(\mathcal{T}, Q)$ is the multiplicity of the total transform of f at Q . Also, $\mathcal{T} - k$ will denote a tree $\mathcal{T}' \cong \mathcal{T}$ so that $w(\mathcal{T}', Q) = \max\{w(\mathcal{T}, Q) - k, 0\}$ for any $Q \in |\mathcal{T}|$. Particularly useful will be the tree

$$(7) \quad \mathcal{T}_P^n(f) := \mathcal{T}_P(f) - 1.$$

Sometimes it will be necessary to compare empty trees. In this case, the conditions $=, \leq, \geq$ are vacuous and hence always satisfied.

(vi) Let $g \in \mathbb{C}\{x, y\}$ be another germ at P . Then one can consider the *restriction of g to $\mathcal{T}_P(f)$* – or $\mathcal{T}_P(f)|_g$ – as another weighted graph isomorphic to $\mathcal{T}_P(f)$, but whose weights are now $\nu_Q(g)$. One can check that the set $I := \{g \in \mathbb{C}\{x, y\} \mid \mathcal{T}|_g \geq \mathcal{T}\}$ defines an ideal. Note that, $\dim_{\mathbb{C}}(\mathbb{C}\{x, y\}/\mathfrak{m}^k) = \binom{k+1}{2}$. Hence

$$(8) \quad \deg(\mathcal{T}) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{I}$$

(vii) Let \mathcal{D} be a plane projective curve, note that its (extended) multiplicity trees do not depend on the equation of \mathcal{D} . Hence they can be denoted as $\mathcal{T}_P(\mathcal{D})$ (resp. $\tilde{\mathcal{T}}_P(\mathcal{D})$). Note that the set of maximal points of $\bigcup_{P \in \text{Sing}(\mathcal{D})} |\tilde{\mathcal{T}}_P(\mathcal{D})|$ is in bijection with $\overline{\mathcal{D}}^{[2]}$.

In case \mathcal{D} is irreducible and has degree d , from (6) and (7) one can rewrite the Noether formula for the genus [4, p. 614] as follows

$$(9) \quad g(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - \sum_{P \in \text{Sing } \mathcal{D}} \deg(\mathcal{T}_P^n(\mathcal{D})).$$

(viii) Let us consider π a resolution of singularities for the plane curve \mathcal{C} . We define the *basic logarithmic ideal sheaf of \mathcal{C} with respect to π* as follows

$$(\mathcal{I}_{\mathcal{C}, \pi}^n)_P := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(\mathcal{C}, \pi)|_h \geq \mathcal{T}_P^n(\mathcal{C}, \pi)\}.$$

If no possible confusion results from the underlying resolution, the sheaf $\mathcal{I}_{\mathcal{C}, \pi}^n$ will be denoted simply by $\mathcal{I}_{\mathcal{C}}^n$.

Remark 3.3. Since π also induces a log resolution of the ideal $C = I(\mathcal{C})$ at any point P , one can also see $\mathcal{I}_{\mathcal{C}}^n$ as the multiplier ideal sheaf of C , that is $\pi_* \mathcal{O}_{\tilde{S}_C}(K_{\tilde{S}_C/\mathbb{P}^2} - F)$, where $C \cdot \mathcal{O}_{\tilde{S}_C} = \mathcal{O}_{\tilde{S}_C}(-F)$.

Analogously, $\mathcal{I}_{\mathcal{C}}^n$ corresponds to the special ideal of quasi-adjunction $\mathcal{A}_0(C)$ as defined in [17, 20].

Let us return to the situation presented at the beginning of this section, where \mathcal{C} is a plane projective curve and π is a resolution of singularities. Let $\mathcal{C}_{ij} := \mathcal{C}_i \cup \mathcal{C}_j$, $d_{ij} := \deg \mathcal{C}_{ij}$, and $g_{ij} := g(\mathcal{C}_{ij})$. We denote by C_{ij} an equation of \mathcal{C}_{ij} (which is C_i if $i = j$ and $C_i C_j$ if $i \neq j$). One has the following.

Proposition 3.4. $\dim H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ij}}^n(d_{ij} - 2)) \geq d_{ij} + g_{ij} - \#\{i, j\}$.

Proof. To ease the notation let us write \mathcal{I} for $\mathcal{I}_{\mathcal{C}_{ij}}^n$. From the exact sequence

$$0 \rightarrow \mathcal{I}(d_{ij} - 2) \rightarrow \mathcal{O}_{\mathbb{P}^2}(d_{ij} - 2) \rightarrow \mathcal{O}/\mathcal{I}(d_{ij} - 2) \rightarrow 0$$

and the fact that $H^\ell(\mathcal{O}(k)) = 0$ for any k and $\ell > 0$, one obtains that

$$(10) \quad h^0(\mathbb{P}^2, \mathcal{I}(d_{ij} - 2)) \geq \binom{d_{ij}}{2} - h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}).$$

In what follows, we will assume $i \neq j$. The case $i = j$ is analogous. First, we will calculate $h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I})$. Note that, by (8), one has

$$h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}) = \sum_{P \in \text{Sing } \mathcal{C}_{ij}} \deg \mathcal{I}_P^n(\mathcal{C}_{ij}) = \sum_{P \in \text{Sing } \mathcal{C}_{ij}} \sum_{Q \in |\mathcal{I}_P(\mathcal{C}_{ij})|} \binom{\nu_Q(\mathcal{C}_{ij})}{2}.$$

Since $\nu_Q(\mathcal{C}_{ij}) = \nu_Q(C_i) + \nu_Q(C_j)$ and $\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab$ one obtains that

$$h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}) = \sum_{P \in \text{Sing } \mathcal{C}_{ij}} \sum_{Q \in |\mathcal{I}_P(\mathcal{C}_{ij})|} \left(\binom{\nu_Q(C_i)}{2} + \binom{\nu_Q(C_j)}{2} + \nu_Q(C_i)\nu_Q(C_j) \right)$$

and finally using (9) one obtains

$$(11) \quad h^0(\mathbb{P}^2, \mathcal{O}/\mathcal{I}) = \binom{d_i - 1}{2} + \binom{d_j - 1}{2} - g_{ij} + d_i d_j.$$

Therefore (10) becomes

$$h^0(\mathbb{P}^2, \mathcal{I}(d_{ij} - 2)) \geq \left[\binom{d_{ij}}{2} - \binom{d_i - 1}{2} - \binom{d_j - 1}{2} - d_i d_j \right] + g_{ij} = (d_i - 1) + (d_j - 1) + g_{ij}.$$

□

Definitions 3.5.

- (i) Let $f \in \mathbb{C}\{x, y\}$ be a holomorphic germ at P and π a resolution of V_f . An ideal $I \subset \mathbb{C}\{x, y\}$ is called a *logarithmic ideal for f at P* if for any germ $h \in I$ the 2-form

$$(12) \quad h \frac{dx \wedge dy}{f}$$

is log-resolution logarithmic at P –with respect to V_f and the resolution π .

- (ii) Let $\mathcal{C} = (C)$ be a curve and let π be a resolution of \mathcal{C} . An ideal sheaf \mathcal{I} on \mathbb{P}^2 is called a *logarithmic ideal sheaf for \mathcal{C}* if its stalks \mathcal{I}_P are logarithmic ideals for the germs C_P of C at any $P \in \mathbb{P}^2$ (with respect to V_{C_P} and the resolution π restricted to a neighborhood of P).

Definition 3.6. Let δ_1 and δ_2 be local branches of f at P . A weighted tree \mathcal{T} is said to be a *logarithmic tree for δ_1 and δ_2* if it satisfies the following properties:

- (i) $\mathcal{T} \cong \mathcal{T}_P(f)$,
(ii) the ideal $I := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h \geq \mathcal{T}\}$ is logarithmic, and

(iii) if $\varphi \in M_I$, where $M_I := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h = \mathcal{T}\} \subset I$, then

$$\left(\text{Res}^{[2]} \varphi \frac{dx \wedge dy}{f} \right)_Q \neq 0$$

if and only if Q is a vertex of the unique subtree $\gamma_P(\delta_1, \delta_2) \subset \mathcal{T}$ joining δ_1 and δ_2 .

Example 3.7. Note that $\mathcal{T}_P^n(f)$ is a logarithmic tree for any two branches δ_1, δ_2 of V_f at P . Moreover, if $\varphi \in \mathbb{C}\{x, y\}$ is a germ at P such that $\psi := \varphi \frac{dx \wedge dy}{f}$ with $\mathcal{T}_P(f)|_\varphi \geq \mathcal{T}_P^n(f)$, then $\psi \in \pi_* W_1^2$, that is, it has weight one and hence $\left(\text{Res}^{[2]} \psi \right)_Q = 0$ for any $Q \in \text{Sing}(\tilde{V}_f)$.

Proof. Note that if $\psi := \varphi \frac{dx \wedge dy}{f}$, then

$$\psi \xleftarrow[y=v]{x=uv} \varphi(uv, v) \frac{d(uv) \wedge dv}{f(uv, v)} = v^{\nu_P(\varphi)+1-\nu_P(f)} \tilde{\varphi}(u, v) \frac{du \wedge dv}{\tilde{f}(u, v)} = \tilde{\psi}$$

in one of the affine charts of the blow-up of P . Since $\nu_P(\varphi) \geq \nu_P(f) - 1$ by hypothesis, the form $\tilde{\psi}$ has poles only along $\tilde{f} = 0$. This argument works inductively, until the result of the blow-up is an irreducible smooth branch. This proves that ψ is log-resolution logarithmic and that it is of type W_1^2 . \square

Theorem 3.8 ([5, Lemma 2.34]). *For any given two local branches δ_1 and δ_2 of f at P there exists a logarithmic tree for δ_1 and δ_2 at P associated with a resolution π of the germ of f at P .*

We will denote such a tree by $\mathcal{T}_P^{\delta_1, \delta_2}(V_f, \pi)$.

In order to construct global forms we will proceed as follows. First, for each irreducible component \mathcal{C}_i of \mathcal{C} , we will order the $d_i = \deg \mathcal{C}_i$ points of \mathcal{C}_i at infinity $\mathcal{C}_i \cap \mathcal{C}_0 = \{P_1^i, \dots, P_{d_i}^i\}$.

Definition 3.9. Let $P \in \mathcal{C}_{ij}$, and let δ_1 (resp. δ_2) be a local branch of the irreducible component \mathcal{C}_i (resp. \mathcal{C}_j) at P . The *ideal sheaf* $\mathcal{I}_{\mathcal{C}_{ij}, \pi}^{\delta_1, \delta_2}$ associated with δ_1 and δ_2 shall be defined as

$$(\mathcal{I}_{\mathcal{C}_{ij}, \pi}^{\delta_1, \delta_2})_Q := \left\{ h \in \mathcal{O}_Q \mid \begin{array}{ll} \mathcal{T}_Q(\mathcal{C}_{ij}, \pi)|_h \geq \mathcal{T}_P^{\delta_1, \delta_2}(\mathcal{C}_{ij}, \pi) & \text{if } Q = P \\ \mathcal{T}_Q(\mathcal{C}_{ij}, \pi)|_h \geq \mathcal{T}_Q(\mathcal{C}_{ij}, \pi) - 2 & \text{if } Q \in \{P_1^i, P_1^j\} \\ \mathcal{T}_Q(\mathcal{C}_{ij}, \pi)|_h \geq \mathcal{T}_Q^n(\mathcal{C}_{ij}, \pi) & \text{otherwise} \end{array} \right\}.$$

Since we have fixed the resolution π , the ideal sheaf $\mathcal{I}_{\mathcal{C}_{ij}, \pi}^{\delta_1, \delta_2}$ will simply be denoted by $\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1, \delta_2}$. A global section s of $\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1, \delta_2}(d)$ shall be called *essential* if $s_Q \in M_{I_Q}$ for every $Q \in \mathbb{P}^2$, where s_Q is the section s localized at Q , $I_Q = (\mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1, \delta_2})_Q$ and M_{I_Q} is as in Definition 3.6(iii).

Analogously to [5, Lemma 3.35] one can prove the following.

Proposition 3.10. $\deg \mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1, \delta_2} = \deg \mathcal{I}_{\mathcal{C}_{ij}}^n + d_{ij} - \#\{i, j\} - 1$.

Therefore Propositions 3.4 and 3.10 imply the following.

Proposition 3.11. $\dim H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1, \delta_2}(d_{ij} - 2)) > g_{ij}$.

One can give a description of a section in such sheaf ideals. In order to do so let us denote by $\gamma_P(\delta_1, \delta_2)$ the minimal subtree in $\mathcal{T}_P(\mathcal{C}_{ij}, \pi)$ (see Definition 3.6) containing δ_1 and δ_2 . We can consider $\gamma_P(\delta_1, \delta_2)$ as a subset of the total transform $\bar{\mathcal{C}}$ of \mathcal{C} (in particular it should contain $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$). We also denote by $v(\gamma_P(\delta_1, \delta_2))$ the set of vertices of $\gamma_P(\delta_1, \delta_2)$.

Proposition 3.12. *Let φ be a section in $H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ij}}^{\delta_1, \delta_2}(d_{ij} - 2))$. Consider the 2-form $\psi_P^{\delta_1, \delta_2} := \varphi \frac{\omega}{\mathcal{C}_0 \mathcal{C}_{ij}}$. One has the following basic properties:*

(1)

$$\left(\text{Res}^{[2]}\psi_P^{\delta_1, \delta_2}\right)_Q = \begin{cases} \pm\lambda & \text{if } Q \in v(\gamma_P(\delta_1, \delta_2)), \\ \pm\lambda' & \text{if } Q \in \{P_1^i, P_1^j\}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The constant $\lambda' \in \mathbb{C}$ satisfies:

$$\lambda' = \begin{cases} \pm\lambda & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $i \neq j$, then $|\lambda| = |\lambda'| \neq 0$ if and only if $\varphi \in M_{\mathcal{I}_{C_{ij}}^{\delta_1, \delta_2}}$ is essential (as in Definition 3.6(iii)).

(3) The signs of the residues described in (1) are such that if $\mathcal{D} \subset \gamma_P(\delta_1, \delta_2) \cup \mathcal{C}_0 \subset \overline{\mathcal{C}}^{[1]}$ is an irreducible component then ${}^2\tilde{R}_0^{[1]}\psi_P^{\delta_1, \delta_2}$ has exactly two poles along \mathcal{D} whose residues are λ and $-\lambda$ so that they add up to zero.

Proof. The proof of Theorem 3.8 is constructive and using such construction one can check parts (1) and (2). Part (3) is a consequence of the commutativity of the generalized residue maps (Theorem 2.9) and the fact that the residues of a meromorphic function on a compact Riemann surface add up to zero. \square

Notation 3.13. For any $P \in \text{Sing}(\mathcal{C} \setminus \mathcal{C}_0)$ and any two local branches δ_1 and δ_2 belonging to the global components \mathcal{C}_i and \mathcal{C}_j respectively. Let $\psi_P^{\delta_1, \delta_2}$ be a log-resolution logarithmic 2-form as constructed in Proposition 3.12 with the extra normalizing condition that $\left(\text{Res}^{[2]}\psi_P^{\delta_1, \delta_2}\right)_{P_1^i} = 1$. The set of all such 2-forms will be denoted by $v_1 := \{\psi_P^{\delta_1, \delta_2}\}_{P, \delta_1, \delta_2}$, the vector space they generate by V_1 , and its projection into $H^2(S_{\mathcal{C}})$ by \tilde{V}_1 .

Remark 3.14. Note that, if $\gamma(\delta_1, \delta_2) := \gamma_P(\delta_1, \delta_2) \cup \mathcal{C}_0 \subset \overline{\mathcal{C}}$ then there is a 1-cycle in \mathcal{C} which results from joining the vertices having non-zero residues. In order to complete this to a set of generators of $H_1(\mathcal{C})$ one needs to consider 1-cycles “at infinity” and those coming from the genus of each irreducible component.

Definition 3.15. For any $P_k^i \in \mathcal{C}_0 \cap \mathcal{C}_i$, $k = 2, \dots, d_i$, the ideal sheaf $\mathcal{I}_{C_i, \pi}^{P_k^i}$ associated with P_k^i shall be defined as

$$(\mathcal{I}_{C_i, \pi}^{P_k^i})_Q := \left\{ h \in \mathcal{O}_Q \mid \begin{array}{ll} \mathcal{T}_Q(C_i, \pi)|_h \geq \mathcal{T}_Q(C_i, \pi) - 2 & \text{if } Q \in \{P_1^i, P_k^i\} \\ \mathcal{T}_Q(C_i, \pi)|_h \geq \mathcal{T}_Q^n(C_i, \pi) & \text{otherwise} \end{array} \right\}.$$

As before, since we have fixed the resolution π , the ideal sheaf $\mathcal{I}_{C_i, \pi}^{P_k^i}$ will simply be denoted by $\mathcal{I}_{C_i}^{P_k^i}$. Again, a global section s of $\mathcal{I}_{C_i}^{P_k^i}(d)$ shall be called *essential* if $\mathcal{T}_Q(C_i, \pi)|_{s_Q} = \mathcal{T}_Q(C_i, \pi) - 2$ for every $Q \in \{P_1^i, P_k^i\}$, where s_Q is the section s localized at Q .

One can also describe such sections in terms of their residues as in Proposition 3.12. Its analogue reads as follows.

Proposition 3.16. Let φ be a section in $H^0(\mathbb{P}^2, \mathcal{I}_{C_i}^{P_k^i}(d_i - 2))$. Consider the 2-form $\psi_\infty^{i, k} := \varphi \frac{\omega}{C_0 C_i}$. One has the following basic properties:

(1)

$$\left(\text{Res}^{[2]}\psi_\infty^{i,k}\right)_Q = \begin{cases} \pm\lambda & \text{if } Q \in \{P_1^i, P_k^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\lambda \neq 0$ if and only if $\varphi \in M_{\mathcal{I}_{C_i}^{P_k^i}}$ is essential (as defined in 3.15).(3) The signs of the residues described in (1) are such that if $\mathcal{D} \in \{\mathcal{C}_i, \mathcal{C}_0\} \subset \overline{\mathcal{C}}^{[1]}$, then ${}^2\tilde{R}_0^{[1]}\psi_\infty^{i,k}$ has exactly two poles along \mathcal{D} whose residues are λ and $-\lambda$ so that they add up to zero.

Proof. The proof follows immediately from 3.7 and the fact that the singularities at infinity are always nodes, hence the local trees are as shown in Figure 1. Therefore $\mathcal{T}_{P_1^i}(C_i) - 2$, imposes no

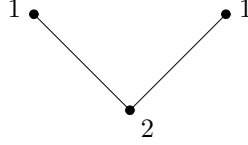


FIGURE 1. Multiplicity tree of a node.

conditions on φ and thus

$$\left(\text{Res}^{[2]}\psi_\infty^{i,k}\right)_{P_1^i} = \pm\varphi(P_1^i) = \pm\lambda.$$

The same argument works for P_k^i , and hence (1)-(2) follow. Finally, part (3) follows from the same ideas as in Proposition 3.12(3). \square

Notation 3.17. For any $P_k^i \in \mathcal{C}_0 \cap \mathcal{C}_i$, $k = 2, \dots, d_i$, let $\psi_\infty^{i,k}$ be a log-resolution logarithmic 2-form as constructed in Proposition 3.16 again with the extra normalizing condition that $\left(\text{Res}^{[2]}\psi_\infty^{i,k}\right)_{P_1^i} = 1$. The set of all such 2-forms will be denoted by $v_\infty := \{\psi_\infty^{i,k}\}_{i,k}$, the vector space they generate by V_∞ , and its projection into $H^2(S_C)$ by \tilde{V}_∞ .

Remark 3.18. As in Remark 3.14, there is a 1-cycle passing through the points P_1^i and P_k^i associated with each 2-form. In the rational arrangement case, these forms generate $H^2(S_C)$ ([5, Theorem 2.46]), which can be proved basically just by checking their residues. In the general case, one still needs to add more 2-forms that have to do with $\ker\left(H^2(S_C) \xrightarrow{\text{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]})\right)$.

By Theorem 2.9 and the exact sequence (5) note that

$$(13) \quad H^2(W_1) \cong H^2(W_1/W_0) \xrightarrow{{}^1\tilde{R}_0^{[1]}} H^1(W_0) = H^1(\overline{\mathcal{C}}^{[1]}).$$

Using the inclusion $\Omega^* \xrightarrow{i^*} W_0^*$ from the complex of global holomorphic forms on $\overline{\mathcal{C}}^{[1]}$ to the complex of global differential forms one has a map $H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1) \xrightarrow{i^1} H^1(\overline{\mathcal{C}}^{[1]})$. Also note that $\dim H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1) = g$. In the following, we will describe generators for $H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1)$.

Proposition 3.19. Let $K_i := \{\psi = \varphi \frac{\omega}{C_i} \mid \varphi \in H^0(\mathbb{P}^2, \mathcal{O}(d_i - 3)), \mathcal{T}_P(C_i)|_\varphi \geq \mathcal{T}_P^{\mathfrak{n}}(C_i)\}$. One has the following properties:

- (1) $K_i \subset W_1^2\left(\mathcal{A}_{\mathbb{P}^2}^{\log}(\mathcal{C})\right)$,
- (2) $\tilde{K}_C \subset \ker\left(H^2(S_C) \xrightarrow{\text{Res}^{[2]}} H^0(\overline{\mathcal{C}}^{[2]})\right)$, where $\tilde{K}_C = \oplus_{i=1}^r \tilde{K}_i$, and \tilde{K}_i is the projection of K_i on $H^2(W_1) \subset H^2(S_C)$, and
- (3) ${}^1\tilde{R}_0^{[1]}\tilde{K}_C = i^1 H^1(\overline{\mathcal{C}}^{[1]}, \Omega^1)$.

Proof. Let us start with part (1). The result is local, and from Example 3.7, it is enough to check it at the points at infinity $\{P_1^i, \dots, P_{d_i}^i\} = \mathcal{C}_0 \cap \mathcal{C}_i$. Since such points are smooth on \mathcal{C}_i by hypothesis, the condition $\mathcal{T}_{P_k^i}(\mathcal{C}_i)|_\varphi \geq \mathcal{T}_{P_k^i}^n(\mathcal{C}_i)$ is vacuous. Hence the local equation of ψ at P_k^i is

$$\varphi(u, v) \frac{du \wedge dv}{v},$$

up to a unit, where $\{v = 0\}$ (resp. $\{u = 0\}$) is the local equation of \mathcal{C}_i (resp. of \mathcal{C}_0). This also proves the first statement of part (2). On the one hand note that

$$\dim K_i \geq \binom{d_i - 1}{2} - \sum_{P \in \text{Sing}(\mathcal{C}_i)} \deg \mathcal{T}_P^n(\mathcal{C}_i) = g_i.$$

On the other hand note that if $\psi_1 + \dots + \psi_r = 0$, $\psi_i \in K_i$, then multiplying by C , one has

$$(14) \quad \varphi_1 C_2 \cdots C_r + C_1 \varphi_2 \cdots C_r + \dots + C_1 C_2 \cdots \varphi_r = 0.$$

Consider $\{Q_1, \dots, Q_{d_1-2}\} \subset \mathcal{C}_1 \setminus (\mathcal{C}_2 \cup \dots \cup \mathcal{C}_r)$ and evaluate such points in (14). One obtains that $\varphi_1(Q_1) = \dots = \varphi_1(Q_{d_1-2}) = 0$. Since \mathcal{C}_1 is irreducible and $\deg \varphi_1 = d_1 - 3$, one obtains that $\varphi_1 = 0$. Proceeding analogously for every φ_i one obtains that $K_C = K_1 \oplus K_2 \oplus \dots \oplus K_r$. This same idea shows that ${}^1\tilde{R}_0^{[1]}\psi = 0$ if and only if $\psi = 0$. Therefore $\dim {}^1\tilde{R}_0^{[1]}\tilde{K}_C \geq g = h^1(\bar{\mathcal{C}}^{[1]}, \Omega^1)$. The inclusion ${}^1\tilde{R}_0^{[1]}\tilde{K}_C \subset i^1 H^1(\bar{\mathcal{C}}^{[1]}, \Omega^1)$ forces i^1 to be an injection and thus, parts (3) and (2) follow for dimension reasons. \square

Remark 3.20. Note that Proposition 3.19(3) implies in particular that cohomology classes outside \tilde{K}_C do not have holomorphic representatives.

Consider $\overline{\tilde{K}_C}$ the conjugate of \tilde{K}_C , which is a subspace of $\ker \left(H^2(S_C) \xrightarrow{\text{Res}^{[2]}} H^0(\bar{\mathcal{C}}^{[2]}) \right)$ of dimension g such that $\tilde{K}_C \oplus \overline{\tilde{K}_C} = \ker \left(H^2(S_C) \xrightarrow{\text{Res}^{[2]}} H^0(\bar{\mathcal{C}}^{[2]}) \right)$.

Corollary 3.21. *Under the above conditions*

$$H^2(S_C) = \tilde{V}_1 \oplus \tilde{V}_\infty \oplus \tilde{K}_C \oplus \overline{\tilde{K}_C}.$$

Proof. From Proposition 3.19 we only need to check the result for $H^2(S_C)/\tilde{K}_C \oplus \overline{\tilde{K}_C}$. The residue map $\text{Res}^{[2]}$ on the quotient is injective. Let us consider $\psi \in V_1$. From Proposition 3.12 $\left(\text{Res}^{[2]}\psi \right)_{P_k^i} = 0$, for any $i = 1, \dots, r$, and any $k = 2, \dots, d_i$. On the other hand, from Proposition 3.16 it is immediate that any 2-form $\psi \in V_\infty$ satisfies that $\sum_{k=1}^{d_i} \left(\text{Res}^{[2]}\psi \right)_{P_k^i} = 0$. Therefore if $\psi \in V_1 \cap V_\infty$, $\left(\text{Res}^{[2]}\psi \right)_{P_k^i} = 0$, $k = 1, \dots, d_i$ and thus $\psi = 0$. \square

Our purpose in what follows is to prove that, for a certain choice of v_1 and v_∞ , the forms Σ , v_1 and v_∞ generate a subalgebra of $H^*(S_C)$ whose relations are satisfied for the 2-forms themselves.

Lemma 3.22. *Let $\psi := \varphi \frac{\omega}{q C_i} \in W_1^2$, where φ , q , and C_i are homogeneous polynomials. If $\left({}^1\tilde{R}_0^{[1]}(\psi) \right)_{\tilde{\mathcal{C}}_i} = 0$, then $\varphi = p C_i$ for some homogeneous polynomial $p \in \mathbb{C}[X, Y, Z]$.*

Proof. Let us consider $P \in \mathcal{C}_i \setminus \text{Sing } \mathcal{C}$, then by (13) and the maximum principle, $\left({}^1R_0^{[1]}(\psi) \right)_{\tilde{\mathcal{C}}_i}|_P = \varphi_P \frac{dz_i}{q_P} = 0$ (that is, at the level of forms, and not only at the level of cohomology classes), where z_i is a local system of homogeneous coordinates around P (note that there is no anti-holomorphic component). Since C_i is irreducible one has $\varphi = p C_i$. \square

Analogously to the rational case, consider $P \in \text{Sing } \mathcal{C}$ and three local branches δ_i , δ_j , and δ_k belonging to the global components \mathcal{C}_i , \mathcal{C}_j , and \mathcal{C}_k respectively.

Proposition 3.23. *Let us assume that $\psi := \varphi \frac{\omega}{C_0 \mathcal{C}_i \mathcal{C}_j \mathcal{C}_k}$ is trivial in $H^2(S_{\mathcal{C}})$ for $\varphi = \varphi_i \mathcal{C}_i + \varphi_j \mathcal{C}_j + \varphi_k \mathcal{C}_k$, where φ is a homogeneous polynomial of degree $d_i + d_j + d_k - 1$. In this case $\varphi = 0$.*

Proof. Since $\text{Res}^{[2]} \psi = 0$, one has that $\psi \in W_1^2$. Furthermore, ${}^1\tilde{R}_0^{[1]}(\psi) = 0$, and hence, by Lemma 3.22, $\varphi = f \mathcal{C}_i \mathcal{C}_j \mathcal{C}_k$, which, by the degree condition, implies $f = 0$. \square

Remark 3.24. Let us consider $P \in \mathcal{C}_i \cap \mathcal{C}_j$ and δ_1 , δ_2 local branches of \mathcal{C}_i and \mathcal{C}_j respectively. Assume that \mathcal{C}_i is a non-rational curve (and analogously if \mathcal{C}_j is a non-rational curve), consider $Q \in \tilde{\mathcal{C}}_i \cap E$ an infinitely near point to P , and assume that $\{u = 0\}$ (resp. $\{v = 0\}$) is the local equation of \mathcal{C}_i (resp. E) at Q . By Proposition 3.12(1) and Notation 3.13 we know that the local equation of $\psi_P^{\delta_1, \delta_2}$ at Q is $\tilde{\varphi} \frac{du \wedge dv}{uv}$ where $\tilde{\varphi}(0) = 1$. We can then choose $\psi_P^{\delta_1, \delta_2}$ such that $\tilde{\varphi}|_{u=0} = 1 + \lambda v + \mathfrak{m}^2$, where λ is defined as follows. Note that $\pi^* \frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j}$ also has poles along E . Therefore locally

$${}^2R_0^{[1]} \pi^* \frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j} \Big|_E = k \frac{dv}{v} + f dv$$

where $k \in \mathbb{C}^*$ and $f \in \mathbb{C}\{v\}$ is holomorphic. Therefore

$$(15) \quad \lambda := \frac{1}{(\delta_1, \delta_2)_P} f(0).$$

Analogously one can construct the 2-forms described in Proposition 3.16. We will denote such a set of closed forms by $v_{\mathcal{C}}$ and the vector space they generate by $E_{\mathcal{C}}$.

Proposition 3.25. *Under the previous conditions, let $P \in \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k$, consider δ_1 , δ_2 , and δ_3 local branches of \mathcal{C}_i , \mathcal{C}_j , and \mathcal{C}_k respectively. If $\psi_P^{\delta_1, \delta_2} = \varphi_P^{\delta_1, \delta_2} \frac{\omega}{C_0 \mathcal{C}_i \mathcal{C}_j}$, $\psi_P^{\delta_2, \delta_3} = \varphi_P^{\delta_2, \delta_3} \frac{\omega}{C_0 \mathcal{C}_j \mathcal{C}_k}$, and $\psi_P^{\delta_3, \delta_1} = \varphi_P^{\delta_3, \delta_1} \frac{\omega}{C_0 \mathcal{C}_k \mathcal{C}_i}$ are forms in $v_{\mathcal{C}}$ constructed as in Remark 3.24, then*

$$\varphi_P^{\delta_1, \delta_2} \mathcal{C}_k + \varphi_P^{\delta_2, \delta_3} \mathcal{C}_i + \varphi_P^{\delta_3, \delta_1} \mathcal{C}_j = 0.$$

Proof. Let us denote by ψ the 2-form $\psi_P^{\delta_1, \delta_2} + \psi_P^{\delta_2, \delta_3} + \psi_P^{\delta_3, \delta_1}$ and by φ the polynomial $\varphi_P^{\delta_1, \delta_2} \mathcal{C}_k + \varphi_P^{\delta_2, \delta_3} \mathcal{C}_i + \varphi_P^{\delta_3, \delta_1} \mathcal{C}_j$. It is immediate that $\text{Res}^{[0]} \psi = 0$ and by the previous construction also ${}^1R_0^{[1]} \psi = 0$. Since $\psi = \varphi \frac{\omega}{C_0 \mathcal{C}_i \mathcal{C}_j \mathcal{C}_k}$, the result follows from Proposition 3.23. \square

Proposition 3.26. *The following equality of 2-forms holds:*

$$(16) \quad \sigma_i \wedge \sigma_j = \sum_{\substack{P \in \text{Sing}(\mathcal{C}_{ij}) \\ \delta_1 \in \Delta_P(\mathcal{C}_i), \\ \delta_2 \in \Delta_P(\mathcal{C}_j)}} (\delta_1, \delta_2)_P \psi_P^{\delta_1, \delta_2} + d_j \sum_{k=2}^{d_i} \psi_{\infty}^{i,k} - d_i \sum_{k=2}^{d_j} \psi_{\infty}^{j,k},$$

where δ_1 (resp. δ_2) runs over the local branches of \mathcal{C}_i (resp. \mathcal{C}_j) at P , and $(\delta_1, \delta_2)_P$ denotes the intersection number.

Proof. Note that $\sigma_i \wedge \sigma_j = \text{Jac}(C_i, C_j, C_0) = \begin{vmatrix} C_{i,X} & C_{i,Y} & C_{i,Z} \\ C_{j,X} & C_{j,Y} & C_{j,Z} \\ C_{0,X} & C_{0,Y} & C_{0,Z} \end{vmatrix} \frac{\omega}{C_0 \mathcal{C}_i \mathcal{C}_j}$. If we denote by ψ_{ij}

the right-hand side of (16) then following the proof of [5, Theorem 2.47], one easily checks that $\text{Res}^{[2]} \sigma_i \wedge \sigma_j = \text{Res}^{[2]} \psi_{ij}$. As a consequence of that $\sigma_i \wedge \sigma_j - \psi_{ij} \in W_1^2$ and hence

one can check ${}^1R_0^{[1]}(\sigma_i \wedge \sigma_j - \psi_{ij})$. By construction (see Remark 3.24) it is easy to see that ${}^1R_0^{[1]}(\sigma_i \wedge \sigma_j - \psi_{ij}) = 0$ (one needs to use that ${}^1R_0^{[1]}(\sigma_i \wedge \sigma_j - \psi_{ij}) = {}^2R_0^{[1]}\sigma_i \wedge \sigma_j - {}^2R_0^{[1]}\psi_{ij}$ and Theorem 2.9), then by Proposition 3.23,

$$(17) \quad \text{Jac}(C_i, C_j, C_0) = \sum_{\substack{P \in \text{Sing}(\mathcal{C}_{ij}) \\ \delta_1 \in \Delta_P(\mathcal{C}_i), \\ \delta_2 \in \Delta_P(\mathcal{C}_j)}} (\delta_1, \delta_2)_P \varphi_P^{\delta_1, \delta_2} + d_j \sum_{k=2}^{d_i} \varphi_\infty^{i,k} C_j - d_i \sum_{k=2}^{d_j} \varphi_\infty^{j,k} C_i$$

follows and hence formula (16) holds. \square

As a consequence of the previous results one obtains a description of the cohomology ring of the complement of a projective plane curve $H^*(S_C)$.

Theorem 3.27. *The cohomology ring H^* of S_C can be decomposed as follows*

$$\tilde{E}_C \oplus \tilde{K}_C \oplus \overline{K}_C$$

where H^* is trivial in degree ≥ 3 , \tilde{K}_C and \overline{K}_C are homogeneous subrings of degree 2 and dimension g each, and \tilde{E}_C can be described as:

- Generated in degrees 1 and 2 by

(G1) (Generators of \tilde{E}_C^1) $\sigma_1, \dots, \sigma_r$,

(G2) (Generators of \tilde{E}_C^2)

$$\{\psi_P^{\delta_1, \delta_2}\}_{P, \delta_1, \delta_2} \cup \{\psi_\infty^{i,k}\}_{i,k}$$

(see Notations 3.13 and 3.17).

- The following is a complete system of relations

(R1) $\psi_P^{\delta_1, \delta_2} + \psi_P^{\delta_2, \delta_1}$,

(R2)

$$\sigma_i \wedge \sigma_j - \sum_{\substack{P \in \text{Sing}(\mathcal{C}_{ij}) \\ \delta_1 \in \Delta_P(\mathcal{C}_i), \\ \delta_2 \in \Delta_P(\mathcal{C}_j)}} (\delta_1, \delta_2)_P \psi_P^{\delta_1, \delta_2} - d_j \sum_{k=2}^{d_i} \psi_\infty^{i,k} + d_i \sum_{k=2}^{d_j} \psi_\infty^{j,k},$$

(R3) $\psi_P^{\delta_1, \delta_2} + \psi_P^{\delta_2, \delta_3} + \psi_P^{\delta_3, \delta_1}$.

\square

Note that the ring structure depends on some local and global data which will be described in what follows. Because of the general condition about the transversal line we will repeat the Definition 2.18 with a slight change in notation.

Let $\tilde{\mathcal{C}} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_r \subset \mathbb{P}^2$, \mathcal{C}_0 a transversal line, $\mathcal{C} := \mathcal{C}_0 \cup \tilde{\mathcal{C}}$, $\mathbf{r} := \{1, \dots, r\}$, $\bar{d} := (d_1, \dots, d_r)$, where $d_i := \deg \mathcal{C}_i$, $\bar{g} := (g_1, \dots, g_r)$, where $g_i := g(\mathcal{C}_i)$, and let $\text{Sing}(\tilde{\mathcal{C}})$ be the collection of singular points of $\tilde{\mathcal{C}}$. For any $P \in \text{Sing}(\tilde{\mathcal{C}})$ let Δ_P be the collection of local branches of $\tilde{\mathcal{C}}$ at P . Consider the following two maps $\phi_P : \Delta_P \rightarrow \mathbf{r}$ and $(\bullet, \bullet)_P : \text{SP}_{\phi_P}^2(\Delta_P) \rightarrow \mathbb{N}$, where $\phi_P(\delta)$ is the global irreducible component containing δ , $\text{SP}_{\phi_P}^2(\Delta_P) := \frac{\Delta_P \times \Delta_P}{\Sigma_2} \setminus \Delta_{\phi_P}$ is the symmetric product of Δ_P outside the ϕ_P -diagonal ($\Delta_{\phi_P} := \{(\delta_1, \delta_2) \mid \phi_P(\delta_1) = \phi_P(\delta_2)\}$), and $(\delta_1, \delta_2)_P$ represents the intersection number of δ_1 and δ_2 at P .

Definition 3.28. The family $W_C := (\mathbf{r}, \text{Sing}(\tilde{\mathcal{C}}), \{\Delta_P, \phi_P, (\bullet, \bullet)_P\}_{P \in \text{Sing}(\tilde{\mathcal{C}})}, \bar{d}, \bar{g})$ is called the *weak combinatorial type* of \mathcal{C} .

Corollary 3.29. *The cohomology ring H^* of $S_{\mathcal{C}}$ only depends on $W_{\mathcal{C}}$ the weak combinatorial type of \mathcal{C} . \square*

Remark 3.30. Corollary 3.29 is also true in the case that the curve does not contain a transversal line – as we have assumed throughout section §3. In this case one can add a transversal line and consider $\mathcal{C} = \tilde{\mathcal{C}} \cup \mathcal{C}_0$. The ring $H^*(S_{\tilde{\mathcal{C}}})$ fits in the following exact sequence

$$0 \rightarrow H^*(S_{\tilde{\mathcal{C}}}) \rightarrow H^*(S_{\mathcal{C}}) \xrightarrow{\pi_{\mathcal{C}_0} \circ \text{Res}^{[1]}} \mathbb{C}_{\mathcal{C}_0} \rightarrow 0,$$

where $\text{Res}^{[1]}$ is the residue defined in Proposition 2.15 and $\pi_{\mathcal{C}_0}$ is the projection of $H^0((\tilde{\mathcal{C}}_0 \cup \overline{\mathcal{C}})^{[1]})$ on the coordinate corresponding to \mathcal{C}_0 .

Example 3.31. Consider the two conics $\mathcal{C}_1 := \{y(y-z) + (x+y)^2 = 0\}$, $\mathcal{C}_2 := \{y(y-z) + (x-y)^2 = 0\}$ and the line $\mathcal{C}_3 := \{y = 0\}$ (see Figure 2). The weak combinatorial type of $\tilde{\mathcal{C}} := \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is $W_{\tilde{\mathcal{C}}} := (\{1, 2, 3\}, S, \{\Delta_P, \phi_P, (\bullet, \bullet)_P\}_{P \in S}, (2, 2, 1), (0, 0, 0))$, where $S := \{P_1, P_2, P'_2, P_3\}$, $\Delta_{P_1} := \{\delta_1^1, \delta_2^1\}$, $\Delta_{P_2} := \{\delta_2^2, \delta_3^2\}$, $\Delta_{P'_2} := \{\delta_1'^2, \delta_3'^2\}$, $\Delta_{P_3} := \{\delta_1^3, \delta_2^3\}$, $\phi_{P_i}(\delta_j^i) := j$, and $(\delta_j^i, \delta_k^i)_{P_i} = i$. The ring $H^*(S_{\tilde{\mathcal{C}}})$ is generated by the 1-forms $\omega_i := 2\sigma_3 - \sigma_i$, $i = 1, 2$ and the 2-forms $\psi_1 := \psi_{P_3}^{\delta_1^3, \delta_2^3} + \psi_{P_2}^{\delta_2^2, \delta_3^2} - \psi_{P'_2}^{\delta_1'^2, \delta_3'^2}$, and $\psi_2 := \psi_{P_1}^{\delta_1^1, \delta_2^1} + \psi_{P_2}^{\delta_2^2, \delta_3^2} - \psi_{P'_2}^{\delta_1'^2, \delta_3'^2}$. The only relation is given by $\omega_1 \wedge \omega_2 = 3\psi_1 + \psi_2$. Hence

$$H^*(S_{\tilde{\mathcal{C}}}) = \langle \omega_1, \omega_2, \psi_1, \psi_2 \mid \omega_1 \wedge \omega_2 = 3\psi_1 + \psi_2 \rangle.$$

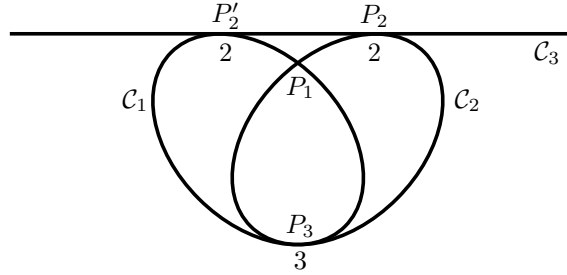


FIGURE 2. Projective realization of \mathcal{C} and multiplicities of intersection

Remark 3.32. As in Definition 3.28 the curve $\tilde{\mathcal{C}} \subset \mathbb{P}^2$ will not be assumed to have a transversal line and usually, we will denote by \mathcal{C} the union of $\tilde{\mathcal{C}}$ and a transversal line. In the future, we will always consider this situation unless otherwise stated.

4. GENERALIZED AOMOTO COMPLEXES AND RESONANCE VARIETIES

4.1. Resonance varieties. Given a curve $\tilde{\mathcal{C}}$ and a holomorphic 1-form on its affine complement $S_{\mathcal{C}} := \mathbb{P}^2 \setminus (\tilde{\mathcal{C}} \cup \mathcal{C}_0)$, say $\omega \in H^1(S_{\mathcal{C}})$ consider the complex $(H^1(S_{\mathcal{C}}), \wedge \omega)$ described by using the algebra structure as follows:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{C} & \xrightarrow{\wedge \omega} & H^1(S_{\mathcal{C}}) & \xrightarrow{\wedge \omega} & H^2(S_{\mathcal{C}}) \rightarrow 0 \\ & & \zeta & \mapsto & \zeta \omega & & \\ & & \alpha & \mapsto & \alpha \wedge \omega. & & \end{array}$$

Definition 4.1. The set

$$\mathcal{R}_i(\mathcal{C}) := \{\omega \in H^1(S_{\mathcal{C}}) \mid \dim H^1(H^1(S_{\mathcal{C}}), \wedge \omega) \geq i\}$$

is called the i -th resonance variety of \mathcal{C} .

Originally in [18] and later on also in [6, 19, 8] (resp. [9]) it is shown that resonance varieties of a plane algebraic curve (resp. of a quasi-projective 1-formal group) can be decomposed into a finite union of linear subspaces. Moreover, in [8] a further description of each irreducible (linear) component is given in terms of pencils containing \mathcal{C} .

Our purpose in this section is to define a purely combinatorial complex $(A_W^*, \wedge \omega)$ whose cohomology jumping loci coincides with $\mathcal{R}_i(\mathcal{C})$. In other words, if

$$\mathcal{V}_i(W) := \{\omega \in A_W^1 \mid \dim H^1(A_W^*, \wedge \omega) \geq i\},$$

there exists a chain of morphisms between A_W^* and $H^*(S_{\mathcal{C}})$ such that $\mathcal{V}_i(W) = \mathcal{R}_i(\mathcal{C})$.

4.2. Generalized Aomoto Complex. Let $W_{\mathcal{C}} = (W, \bar{d}, \bar{g})$ be the weak combinatorial type of \mathcal{C} as in Definition 3.28 and consider:

- (i) $A_W^0 := \mathbb{C}$,
 - (ii) $A_W^1 := \bigoplus_{i \in \mathbf{r}} \sigma_i \mathbb{C}$,
 - (iii) $A_W^2 := \bigoplus_{P \in S} \frac{A_P \wedge A_P}{I_P}$, where $A_P = \bigoplus_{\delta \in \Delta_P} \psi_P^\delta \mathbb{C}$ and
- $$(18) \quad I_P := \{\psi_P^{\delta_1} \wedge \psi_P^{\delta_2} + \psi_P^{\delta_2} \wedge \psi_P^{\delta_3} + \psi_P^{\delta_3} \wedge \psi_P^{\delta_1}\} + \{\psi_P^{\delta_1} \wedge \psi_P^{\delta_2} \mid \#\phi_P(\Delta_P) = 1\} \subset A_P \wedge A_P.$$
- (iv) $A_W^i = 0$ ($i \geq 3$).

Finally we turn A_W^* into a graded algebra by means of the following product:

$$(19) \quad \sigma_i \wedge \sigma_j := \sum_{\substack{P \in S, \\ \phi_P(\delta_1) = i, \\ \phi_P(\delta_2) = j}} (\delta_1, \delta_2)_P \psi_P^{\delta_1} \wedge \psi_P^{\delta_2}.$$

Definition 4.2. Given an element of weight one, say $\omega \in A_W^1$, one can consider the complex $(A_W^*, \wedge \omega)$ described by using the algebra structure as follows:

$$\begin{array}{ccccccc} 0 & \rightarrow & A_W^0 & \xrightarrow{\wedge \omega} & A_W^1 & \xrightarrow{\wedge \omega} & A_W^2 \rightarrow 0 \\ & & \zeta & \mapsto & \zeta \omega & & \\ & & & & \alpha & \mapsto & \alpha \wedge \omega. \end{array}$$

Such a complex is defined as the *Generalized Aomoto Complex of W* .

Remark 4.3. In order to describe a matrix for the homomorphism $A_W^1 \xrightarrow{\wedge \omega} A_W^2$ we need a basis for A_W^2 (note that $\{\sigma_i\}_{i \in \mathbf{r}}$ is a basis for the vector space A_W^1). We will proceed as follows:

- (1) for any point $P \in S$ one can choose a local branch $\delta_P^0 \in \Delta_P$. The element δ_P^0 will be referred to as the *preferred branch of W at P* .
- (2) Using this notation note that the set of vectors

$$(20) \quad \{\psi_P^{\delta_P^0} \wedge \psi_P^\delta\}_{\substack{P \in S, \\ \#\phi_P(\Delta_P) > 1}}$$

is a basis of the vector space A_W^2 .

As usual, one can define the *cohomology jumping loci* of such a family of complexes parametrized by A_W^1 as

$$\mathcal{V}_i(W) := \{\omega \in A_W^1 \mid \dim H^1(A_W^*, \wedge \omega) \geq i\}.$$

One has the following.

Theorem 4.4. *By means of the canonical isomorphism $A_W^1 \rightarrow H^1(S_C)$, the complexes $(A_W^*, \wedge \omega)$ and $(H^*(S_C), \wedge \omega)$ are in the same quasi-isomorphism class. Therefore, one can identify $\mathcal{V}_i(W)$ and $\mathcal{R}_i(C)$.*

Proof. Consider the subalgebra \tilde{E}_C^* of $H^*(S_C)$ as described in Theorem 3.27. Note that $\tilde{E}_C^1 = H^1(S_C)$, and hence for any $\omega \in \tilde{E}_C^1$ one can consider the complex $(\tilde{E}_C^*, \wedge \omega)$ and its cohomology jumping loci $\mathcal{V}_i(\tilde{E}_C) := \{\omega \in \tilde{E}_C^1 \mid \dim H^1(\tilde{E}_C^*, \wedge \omega) \geq i\}$. Note that $\mathcal{V}_i(\tilde{E}_C) = \mathcal{R}_i(C)$.

On the other hand consider \tilde{E}_C^*/I^0 , where $I^0 \subset \tilde{E}_C^2$ is generated by $\{\psi_P^{\delta_1, \delta_2} \mid P \in \text{Sing}(C), \# \phi_P(\Delta_P) = 1\}$ and the following morphism

$$\begin{aligned} \alpha : \quad A_W^* &\rightarrow \tilde{E}_C^*/I^0 \\ \sigma_k &\mapsto \sigma_k \\ \psi_P^\delta \wedge \psi_P^{\delta'} &\mapsto \psi_P^{\delta, \delta'} + \frac{1}{d_i} \left(\sum_{k=2}^{d_i} \psi_\infty^{k, i} \right) - \frac{1}{d_j} \left(\sum_{k=2}^{d_j} \psi_\infty^{k, j} \right), \end{aligned}$$

where $i := \phi_P(\delta)$ and $j := \phi_P(\delta')$.

First of all note that α is well defined, since $\alpha(\psi_P^{\delta_1} \wedge \psi_P^{\delta_2}) + \alpha(\psi_P^{\delta_2} \wedge \psi_P^{\delta_3}) + \alpha(\psi_P^{\delta_3} \wedge \psi_P^{\delta_1}) = 0$ and that

$$\begin{aligned} \alpha(\sigma_i \wedge \sigma_j) &= \sum_{\substack{P \in S, \\ \phi_P(\delta_1) = i, \\ \phi_P(\delta_2) = j}} (\delta_1, \delta_2)_P \alpha(\psi_P^{\delta_1} \wedge \psi_P^{\delta_2}) = \\ &= d_j \sum_{k=2}^{d_i} \psi_\infty^{k, i} - d_i \sum_{k=2}^{d_j} \psi_\infty^{k, j} + \sum_{\substack{P \in S, \\ \phi_P(\delta_1) = i, \\ \phi_P(\delta_2) = j}} (\delta_1, \delta_2)_P \psi_P^{\delta_1, \delta_2} = \alpha(\sigma_i) \wedge \alpha(\sigma_j), \end{aligned}$$

where the first equality is true by (19), the second is due to the Bezout Theorem, and the last one is a consequence of the construction of the product in E_C^* as shown in Theorem 3.27(R2).

A straightforward computation shows that α induces quasi-isomorphisms between $(A^*, \wedge \omega)$ and $(\tilde{E}_C^*/I^0, \wedge \alpha(\omega))$. Also, the projection $\tilde{E}_C^* \rightarrow \tilde{E}_C^*/I^0$ induces a quasi-isomorphism. Hence $\mathcal{V}_i(W) \stackrel{\alpha^*}{\cong} \mathcal{V}_i(\tilde{E}_C)$ and thus the result follows. \square

As a consequence of Theorem 4.4 one has the following.

Corollary 4.5. *If $C \subset \mathbb{P}^2$ is a curve and $W_C = (W, \bar{d}, \bar{g})$ its weak combinatorial type, then its resonance varieties $\mathcal{R}_i(C)$ only depend on W . \square*

5. THE STRUCTURE OF CHARACTERISTIC VARIETIES AND COMBINATORIAL PENCILS

Let $\rho : \pi_1(S_C) \rightarrow \text{Aut}(\mathbb{C}, \mathbb{C}) = \mathbb{C}^*$ be a local system of rank one. This defines a rank-one vector bundle \mathcal{V}_ρ on S_C . The cohomology $H^*(S_C, \mathcal{V}_\rho)$ is called *twisted cohomology of S_C with coefficients on the local system ρ* . There is a natural stratification of the space of rank-one local systems on S_C given by:

$$\Sigma_i(C) := \{\rho \in H^1(S_C; \mathbb{C}^*) \mid \dim H^1(S_C, \mathcal{V}_\rho) \geq i\}.$$

Each $\Sigma_i(C)$ is called the *i-th characteristic variety of C* . Characteristic varieties of plane algebraic curves were first defined and studied by A.Libgober in [18] both from the point of view of algebraic conditions for the existence of components and in their connection with resonance varieties. In that foundational paper (see also [6, 19]) the following equality is proved:

$$(\mathcal{R}_i(\mathcal{C}), 0) = (\Sigma_i(\mathcal{C}), \mathbb{1}),$$

as germs, where $\mathbb{1}$ is the trivial character on $S_{\mathcal{C}}$.

Recently, in [9] Dimca-Papadima-Suciu proved that this formula is also true for general 1-formal groups.

From Corollary 4.5 we have determined that $\mathcal{R}_i(\mathcal{C})$ is combinatorial. Our purpose here is to study more closely the combinatorial structure of $(\Sigma_i(\mathcal{C}), \mathbb{1})$. In order to do so we define the concept of combinatorial pencil for projective plane curves.

Definition 5.1. Let $W_{\mathcal{C}} = (W, \bar{d}, \bar{g})$ be an abstract weak combinatorial type, $\bar{m} := (m_i)_{i \in \mathbf{r}}$ a list of integers, $\mathcal{F} = \{F_1, \dots, F_k\}$, $k \geq 3$ a partition of \mathbf{r} . We say that (\mathcal{F}, \bar{m}) is a *combinatorial pencil of $\tilde{\mathcal{C}}$* if

- (1) $d_{\mathcal{F}} = \sum_{i \in F_j} m_i d_i$ for any $j \in \{1, \dots, k\}$, and
- (2) for any $P \in \text{Sing}(\mathcal{C})$ one of the following two conditions is satisfied:
 - (a) either $\phi_P(\Delta_P) \subset F_i$ for a certain $i = 1, \dots, k$,
 - (b) or

$$k_{P,i} = \sum_{\substack{\phi_P(\delta_1) = i \\ \phi_P(\delta_2) \in F_j}} m_{\phi_P(\delta_2)} (\delta_1, \delta_2)_P$$

is independent of $j \in \{1, \dots, k\}$ for any $i \notin F_j$.

The points $P \in \text{Sing}(\tilde{\mathcal{C}})$ satisfying (2b) will be called the *base points of the combinatorial pencil* and each $F_i \in \mathcal{F}$ will be called a *fiber*.

Our purpose in what follows is to prove that combinatorial pencils of $\tilde{\mathcal{C}}$ are in fact algebraic pencils. We will start by checking that combinatorial pencils contribute to the resonance varieties of \mathcal{C} .

Proposition 5.2. Let $W_{\mathcal{C}} := (W, \bar{d}, \bar{g})$ be the weak combinatorics of \mathcal{C} and (\mathcal{F}, \bar{m}) a combinatorial pencil of $\tilde{\mathcal{C}}$, $k := \#\mathcal{F}$. Then $\mathcal{V}_{k-1}(W)$ contains a $(k-1)$ -dimensional linear subspace defined by (\mathcal{F}, \bar{m}) .

Proof. Define $\omega := \sum_{i=1}^k \lambda_i \sigma_{F_i}$, where $\sigma_{F_i} := \sum_{j \in F_i} m_j \sigma_j$ and $\sum_{i=1}^k \lambda_i = 0$ and $\sigma_i \in A_W^1$. Let us prove that $(\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \in \ker(A_W^1 \xrightarrow{\wedge \omega} A_W^2)$. Note that

$$(\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega = (\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \left(\omega_{i_1, i_2, i_3}^{\lambda_{i_1}, \lambda_{i_2}} + \omega_{i_3, i_4, \dots, i_k}^{\lambda_{i_1} + \lambda_{i_2} + \lambda_{i_3}, \lambda_{i_4}, \dots, \lambda_{i_{k-1}}} \right),$$

where $\omega_{j_1, j_2, \dots, j_s}^{\alpha_1, \alpha_2, \dots, \alpha_{s-1}} := \alpha_1 \sigma_{F_{j_1}} + \alpha_2 \sigma_{F_{j_2}} + \dots + \alpha_{s-1} \sigma_{F_{j_{s-1}}} - (\alpha_1 + \dots + \alpha_{s-1}) \sigma_{F_{j_s}}$. The result is a consequence of the following:

- (i) $(\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3}^{\lambda_1, \lambda_2} = 0$,
- (ii) $(\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{j_1, j_2, \dots, j_s}^{\alpha_1, \alpha_2, \dots, \alpha_s} = 0$, if $\{i_1, i_2\} \cap \{j_1, j_2, \dots, j_s\} = \emptyset$.

We will prove (i) in detail. Property (ii) can be proved analogously.

In order to check such an equality we just need to prove that the coefficients of $(\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3}^{\lambda_1, \lambda_2}$ at every element of the basis considered in (20) are zero. In general we will denote the coefficient of $\psi \in A_W^2$ at the element $\psi_P^{\delta_P^0} \wedge \psi_P^{\delta}$ by $\psi_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta}}$.

In order to keep the geometrical interpretation we will assume $\mathbf{r} := \{\mathcal{C}_1, \dots, \mathcal{C}_r\}$. Note that if $\mathcal{C}_{\alpha} \notin F_i \cup F_j$, and $\delta \in \Delta_P$ with $\phi_P(\delta) = \mathcal{C}_{\alpha}$ then $(\sigma_{F_i} \wedge \sigma_{F_j})_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta}} = 0$.

Second, we will check the coefficients of the vectors of type $\psi_P^{\delta_P^0} \wedge \psi_P^\delta$, where P is not a base point of the combinatorial pencil and δ_P^0 is a preferred branch at P . Note that in this case $\phi_P(\Delta_P) \subset F_i$ for a certain fiber F_i and hence the only products that contribute to this coefficient are those of type $\sigma_\alpha \wedge \sigma_\beta$, where $\mathcal{C}_\alpha, \mathcal{C}_\beta \in F_i$. We consider two cases:

- (1) If $i \notin \{i_1, i_2, i_3\}$ or $i = i_3$, then it is immediate that

$$\left((\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3} \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^\delta} = 0.$$

- (2) If $i = i_1$ (analogously $i = i_2$), then

$$\left((\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3} \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^\delta} = \sigma_{F_{i_1}} \wedge \sigma_{F_{i_1}} = 0.$$

Finally we will check the coefficients of the vectors of type $\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}$, where P is a base point of the combinatorial pencil and δ_P^0 a preferred branch at P (see Remark 4.3). For simplicity we will assume \mathcal{C}_1 is the irreducible component of \mathcal{C} containing the local preferred branch δ_P^0 and $\phi_P(\delta') = \mathcal{C}_2$. Note that according to the definition of A^2 (see (18)) one has $\psi_P^{\delta_1} \wedge \psi_P^{\delta_2} = \psi_P^{\delta'_1} \wedge \psi_P^{\delta'_2}$ as long as $\phi_P(\delta_i) = \phi_P(\delta'_i)$, $i = 1, 2$. Therefore note that

$$(\sigma_\alpha \wedge \sigma_\beta)_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} = \begin{cases} \sum_{\substack{\phi_P(\delta_1) = \mathcal{C}_\alpha \\ \phi_P(\delta_2) = \mathcal{C}_\beta}} (\delta_1, \delta_2)_P & \text{if } \alpha \neq \beta = 2 \\ - \sum_{\substack{\phi_P(\delta_1) = \mathcal{C}_\alpha \\ \phi_P(\delta_2) = \mathcal{C}_\beta}} (\delta_1, \delta_2)_P & \text{if } 2 = \alpha \neq \beta \\ 0 & \text{otherwise.} \end{cases}$$

Hence one has

$$(21) \quad (\sigma_{F_i} \wedge \sigma_{F_j})_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} = \begin{cases} m_2 \sum_{\substack{\phi_P(\delta_1) \in F_i \\ \phi_P(\delta_2) = \mathcal{C}_2}} m_{\phi_P(\delta_1)} (\delta_1, \delta_2)_P = m_2 k_{P,2} & \text{if } \mathcal{C}_2 \in F_j, i \neq j, \\ -m_2 \sum_{\substack{\phi_P(\delta_1) = \mathcal{C}_2 \\ \phi_P(\delta_2) \in F_j}} m_{\phi_P(\delta_2)} (\delta_1, \delta_2)_P = -m_2 k_{P,2} & \text{if } \mathcal{C}_2 \in F_i, i \neq j, \\ 0 & \text{if } i = j, \text{ or } \mathcal{C}_2 \notin F_i \cup F_j. \end{cases}$$

We will consider several cases:

- (1) If $\mathcal{C}_2 \in F_{i_1}$ (analogously if $\mathcal{C}_2 \in F_{i_2}$), then

$$\begin{aligned} & \left((\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3} \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} = \\ & = \left(\sigma_{F_{i_1}} \wedge (\lambda_2 \sigma_{F_{i_2}} - (\lambda_1 + \lambda_2) \sigma_{F_{i_3}}) \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} - \left(\sigma_{F_{i_2}} \wedge (\lambda_1 \sigma_{F_{i_1}} - (\lambda_1 + \lambda_2) \sigma_{F_{i_3}}) \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} = \\ & = (-\lambda_2 m_2 k_{P,2} + (\lambda_1 + \lambda_2) m_2 k_{P,2}) - \lambda_1 m_2 k_{P,2} = 0. \end{aligned}$$

- (2) If $\mathcal{C}_2 \in F_{i_3}$, then

$$\begin{aligned} & \left((\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3} \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} = -(\lambda_1 + \lambda_2) \left((\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \sigma_{i_3} \right)_{\psi_P^{\delta_P^0} \wedge \psi_P^{\delta'}} = \\ & = -(\lambda_1 + \lambda_2) m_2 (-k_{P,2} + k_{P,2}) = 0. \end{aligned}$$

□

One has the following result, which is a combinatorial version of the celebrated Max Noether Fundamental Theorem –see for instance [25, 13].

Theorem 5.3. *Let $\tilde{\mathcal{C}} \subset \mathbb{P}^2$ be a curve, $W_{\mathcal{C}} = (W, \bar{d}, \bar{g})$ its weak combinatorial type, (\mathcal{F}, \bar{m}) a combinatorial pencil of $\tilde{\mathcal{C}}$, and define $D_i := \prod_{C_j \in F_i} C_j^{m_j}$, $i = 1, \dots, k$. Then $d := \sum_{i \in F_j} m_i d_i$ is independent of j for any $j \in \{1, \dots, k\}$ and D_1, \dots, D_k generate a pencil of curves of degree d .*

Proof. If $k = 1, 2$ there is nothing to prove. We will hence assume that $k \geq 3$. By Proposition 5.2 $\mathcal{V}_{k-1}(W)$ contains the $(k-1)$ -dimensional vector space $\langle \omega_{1, \dots, k}^{\lambda_1, \dots, \lambda_{k-1}} \mid \lambda_i \in \mathbb{C} \rangle$. Since $\mathcal{V}_i(W) = \mathcal{R}_i(\mathcal{C})$ by Theorem 4.4, and $k-1 \geq 2$ the result is a consequence of [8, Theorem 4.1]. \square

Proof. We will prove (i) in detail. Property (ii) can be proved analogously.

In order to check such equality we just need to prove that the coefficients of $(\sigma_{F_{i_1}} - \sigma_{F_{i_2}}) \wedge \omega_{i_1, i_2, i_3}^{\lambda_1, \lambda_2}$ at every element of the basis considered in (20) are zero. In general we will denote the coefficient of $\psi \in A_W^2$ at the element $\psi_P^{\delta_P^0} \wedge \psi_P^\delta$ by $\psi_{\psi_P^{\delta_P^0} \wedge \psi_P^\delta}$.

Note that if $P \in \mathcal{C}_\alpha \notin F_i \cup F_j$, then $(\sigma_{F_i} \wedge \sigma_{F_j})_{\psi_P^{\delta_P^0} \wedge \psi_P^\delta} = 0$.

\square

Therefore the local components of the characteristic varieties at the origin can be determined from the existence of a combinatorial pencil of curves containing the components of the given curve $\tilde{\mathcal{C}}$.

In order to make a precise statement we need some terminology. Note that given any combinatorial pencil (\mathcal{F}, \bar{m}) of $\tilde{\mathcal{C}}$, one can associate with it a *dual incidence graph* whose vertices are in bijection with \mathbf{r} and having as many edges joining i and j as $\#\{P \in S \mid \phi_P^{-1}(i) \cap \phi_P^{-1}(j) \neq \emptyset\}$.

Definition 5.4. A combinatorial pencil $\mathcal{P} := (\mathcal{F}, \bar{m})$ is called *primitive* if each connected component of the dual incidence graph after removing the edges corresponding to the base points of \mathcal{P} contains the vertices of exactly one fiber.

Remark 5.5. Note that any combinatorial pencil can be refined as to obtain a primitive combinatorial pencil. Therefore unless otherwise stated all the combinatorial pencils from now on are assumed to be primitive. Also note that in Theorem 5.3 a primitive combinatorial pencil gives rise to a primitive pencil (one whose fibers after resolution are connected).

Definition 5.6. We say a combinatorial pencil (\mathcal{F}, \bar{m}) *contains* a curve \mathbf{r}_1 if $\mathbf{r}_1 \subset \mathbf{r}$ is a union of fibers of (\mathcal{F}, \bar{m}) . More generally, if (\mathcal{F}, \bar{m}) contains a subset of \mathbf{r}_1 we say that (\mathcal{F}, \bar{m}) *partially contains* \mathbf{r}_1 .

The following result generalizes the one for line arrangements obtained independently by M.Á.Marco [23] and M.Falk-S.Yuzvinsky [12].

Corollary 5.7. *Let $\tilde{\mathcal{C}}$ be a projective plane curve and consider $\mathcal{C} := \mathcal{C}_0 \cup \tilde{\mathcal{C}}$, where \mathcal{C}_0 is a transversal line. There is a one-to-one correspondence between irreducible components (resp. essential components) of the characteristic variety $\Sigma_k(\mathcal{C})$ containing the trivial character and combinatorial pencils partially containing $\tilde{\mathcal{C}}$ (resp. containing $\tilde{\mathcal{C}}$) with $k+1$ fibers.*

Proof. It is a consequence of Theorem 5.3 and [8, Theorem 4.1]. \square

6. FORMALITY OF COMPLEMENTS TO PROJECTIVE PLANE CURVES

All the basic definitions of minimal algebras, minimal models, homotopy, etc required in the definition of formality and in the theory of homotopy theory of algebras can be found for instance in any of the foundational papers [7, 24, 26].

Definition 6.1. Two graded differential algebras (A, d_A) and (B, d_B) are called *quasi-isomorphic* if there exists a morphism of graded algebras $f : A \rightarrow B$ such that the induced morphism $f^* : H^*(A, d_A) \rightarrow H^*(B, d_B)$ is an isomorphism. Note that “being quasi-isomorphic” is not an equivalence relation. We will refer to the *quasi-isomorphism class* of a graded differential algebra as the minimal equivalence class generated by the quasi-isomorphism relation.

A minimal differential graded algebra is called *formal* if it is quasi-isomorphic to its cohomology algebra using a zero differential. A differential graded algebra is called *formal* if its minimal model is formal. Finally, a complex space X is called *formal* if the algebra of differential forms $(\mathcal{E}(X), d)$ is formal.

The concept of formal algebra is well defined since any differential graded algebra has a unique (up to homotopy) minimal model (c.f. [26, Section §5]). Also note that a minimal model for (A, d_A) consists of a minimal algebra $(\mathcal{M}(A), d_{\mathcal{M}(A)})$ plus a quasi-isomorphism $\mathcal{M}(A) \rightarrow A$. Therefore, if one finds a quasi-isomorphism between $(\mathcal{E}(X), d)$ and $(H(X), 0)$ then X is formal. Moreover, if X is a smooth complex variety and \overline{X} is a completion of X by a simple normal crossing divisor, then the minimal model of $\mathcal{E}(X)$ and the minimal model of $\mathcal{A}_{\overline{X}}(\log\langle D \rangle)$ are isomorphic (c.f. [24, Section §6]).

Let $\mathcal{C} \subset \mathbb{P}^2$ as in Section 2.2.

Lemma 6.2. *There is a decomposition $\mathcal{A}_{\mathbb{P}^2}^{\log}(\mathcal{C}) = \mathbb{C} \oplus \langle \frac{dC_i}{C_i} \rangle \oplus E_{\mathcal{C}} \oplus K_{\mathcal{C}} \oplus \mathcal{E}$ of (log-resolution logarithmic) differential forms on \mathbb{P}^2 , where \mathcal{E} is the kernel of all the residue maps ${}^\ell R_k^{[m]}$ (see (2)).*

Proof. There is an obvious surjective morphism from $\mathcal{A}_{\mathbb{P}^2}^{\log}(\mathcal{C})$ to the right hand side given by the residues. The key to proving injectivity is Lemma 3.22 where φ need not be a polynomial. \square

Theorem 6.3. *There is a well-defined quasi-isomorphism $\mathcal{A}_{S_{\mathcal{C}}}^*(\log\langle \overline{\mathcal{C}} \rangle) \rightarrow H^*(S_{\mathcal{C}})$ given by sending $W_0^*(\overline{S}_{\mathcal{C}})$ to zero.*

Proof. Note that the relations (17) and (3.25) also occur in $H^*(S_{\mathcal{C}})$, therefore the map is well-defined. \square

As a consequence of the discussion at the beginning of this section one has the following.

Theorem 6.4. *The complement of a plane projective curve $S_{\mathcal{C}}$ is a formal space.*

Remark 6.5. Theorem 6.4 is the global version of the formality of algebraic links proved by Durfee-Hain in [10] and obtained independently by A. Macinic in [22]. This result is a consequence of a more general fact proved in that paper: a 2-complex X which is 1-formal is also a formal space.

The 1-minimal model $\mathcal{M}_1(A)$ of a differential graded algebra (A, d_A) is the subalgebra generated by the degree 1 part in $\mathcal{M}(A)$. Then a space X is 1-formal if $\mathcal{M}_1(\mathcal{E}(X), d)$ is quasi-isomorphic to $\mathcal{M}_1(H^*(X), 0)$. This condition can be restated in terms of the fundamental group as follows. A finitely presented group G is 1-formal if its Malcev completion is filtered isomorphic to its holonomy Lie algebra, completed with respect to bracket length. Fundamental groups of complements to algebraic plane curves are known to be 1-formal, (see [16] and [24]).

7. EXAMPLES

7.1. Weak combinatorics does not determine classical combinatorics. Let $\ell_0 := \{x = 0\}$, $\ell_1 := \{y = 0\}$, and $\ell_2 := \{z = 0\}$ be three lines in general position and consider:

- (1) $\tilde{\mathcal{C}}_1 := \{(x-y)^2 - (x+y)z = 0\}$ a conic tangent to ℓ_2 at $(1, 1, 0)$ and passing through $\ell_0 \cap \ell_1$,
- (2) $\tilde{\mathcal{C}}_2^{(1)} := \{x - y + z = 0\}$ the line passing through $\ell_0 \cap \tilde{\mathcal{C}}_1$ and $\ell_2 \cap \tilde{\mathcal{C}}_1$, and
- (3) $\tilde{\mathcal{C}}_2^{(2)} := \{3x - y + z = 0\}$ the line tangent to $\tilde{\mathcal{C}}_1$ at $\ell_0 \cap \tilde{\mathcal{C}}_1$.

The Cremona transformation based on ℓ_0, ℓ_1 , and ℓ_2 transforms $\tilde{\mathcal{C}}^{(k)} := \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2^{(k)}$ into $\mathcal{C}^{(k)}$, a union of a cuspidal cubic \mathcal{C}_1 and a conic $\mathcal{C}_2^{(k)}$. Note that $\mathcal{C}^{(k)}$ has three singular points $\{P_1, P_2, P_3\}$, where $\Delta_{\mathcal{C}^{(k)}, P_i} := \{\delta_i^1, \delta_i^{2,k}\}$, $\phi_{\mathcal{C}^{(k)}, P_i}(\delta_i^1) = \mathcal{C}_1$, $\phi_{\mathcal{C}^{(k)}, P_i}(\delta_i^{2,k}) = \mathcal{C}_2^{(k)}$, $(\delta_i^1, \delta_i^{2,1})_{\mathcal{C}^{(1)}, P_i} = i$, and $(\delta_i^1, \delta_i^{2,2})_{\mathcal{C}^{(2)}, P_i} = \sigma_{(2,3)}(i)$ (where $\sigma_{(2,3)}(i)$ represents the permutation $(2, 3)$ applied to i). Figure 3 represents the singular points of $\mathcal{C}^{(k)}$, the local branches at those points (solid line for \mathcal{C}_1 and broken line for $\mathcal{C}_2^{(k)}$), and the multiplicity of intersection in brackets. Note that the bijection φ_{Sing} that permutes P_2 and P_3 induces an equivalence between $W_{\mathcal{C}^{(1)}}$ and $W_{\mathcal{C}^{(2)}}$ ($\varphi_{\mathbf{r}_{\mathcal{C}^{(1)}}}$ and φ_{P_i} are forced by their compatibility with the degrees and with φ_{Sing}). The combinatorial types $K(\mathcal{C}^{(1)})$ and $K(\mathcal{C}^{(2)})$ cannot be equivalent since the sets of topological types do not coincide, that is, $\Sigma_{\mathcal{C}^{(1)}}^{\text{top}} \neq \Sigma_{\mathcal{C}^{(2)}}^{\text{top}}$.

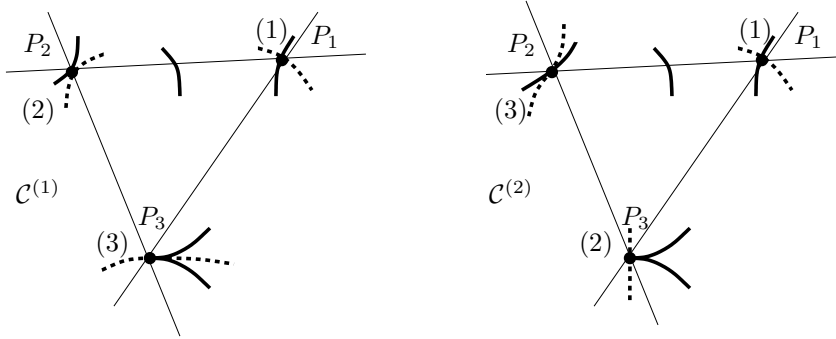


FIGURE 3. Singularities of $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ respectively.

7.2. The complexes $(H^*(S_{\mathcal{C}}), \wedge \omega)$ and $(A_W^*, \wedge \omega)$. An easy example to exhibit the difference between $(H^*(S_{\mathcal{C}}), \wedge \omega)$ and $(A_W^*, \wedge \omega)$ can be given as follows. Consider the nodal cubic by the zeroes of the polynomial $C_2 := zy^2 - x^2(x - z)$, a smooth cubic intersecting C_2 at an inflexion point P with multiplicity 9, say $C_3 := zy^2 - x^2(x - z) + z^3$, and the tangent line at P given by $C_1 := z$. Note that $\mathcal{C} := C_1 \cup C_2 \cup C_3$ has two singular points $P := [0 : 1 : 0]$ and $Q := [0 : 0 : 1]$. Therefore its weak combinatorial type $W_{\mathcal{C}} = (W, \bar{d}, \bar{g})$ can be described as $\mathbf{r} := \{1, 2, 3\}$, $S := \{P, Q\}$, $\Delta_P := \{\delta_{P,1}, \delta_{P,2}, \delta_{P,3}\}$, $\Delta_Q := \{\delta_Q, \delta'_Q\}$, $\phi_P(\delta_{P,i}) := i$, $\phi_P(\delta_Q) = \phi_P(\delta'_Q) = 2$, and $(\delta_{P,1}, \delta_{P,i})_P = 3$, $(\delta_{P,2}, \delta_{P,3})_P = 9$, $(\delta_{Q,1}, \delta'_{Q,1})_Q = 1$. The list of degrees is $\bar{d} := (1, 3, 3)$ and the list of genera is $\bar{g}_{\mathcal{C}} := (0, 0, 1)$ and hence $H^*(S_{\mathcal{C}})$ can be described as follows:

$$\begin{aligned}
H^1(S_{\mathcal{C}}) &:= \langle \sigma_1, \sigma_2, \sigma_3 \rangle_{\mathbb{C}}, \\
H^2(S_{\mathcal{C}}) &:= \langle \psi_P^{1,2}, \psi_P^{2,3}, \psi_P^{3,1}, \psi_Q^{1,1'}, \psi_{\infty}^{2,1}, \psi_{\infty}^{2,2}, \psi_{\infty}^{3,1}, \psi_{\infty}^{3,2}, \xi_1, \bar{\xi}_1 : \psi_P^{1,2} + \psi_P^{2,3} + \psi_P^{3,1} = 0 \rangle_{\mathbb{C}}, \\
\sigma_1 \wedge \sigma_2 &= 3\psi_P^{1,2} - \psi_{\infty}^{2,1} - \psi_{\infty}^{2,2}, \\
\sigma_1 \wedge \sigma_3 &= -3\psi_P^{3,1} - \psi_{\infty}^{3,1} - \psi_{\infty}^{3,2}, \\
\sigma_2 \wedge \sigma_3 &= 9\psi_P^{2,3} + 3\psi_{\infty}^{2,1} + 3\psi_{\infty}^{2,2} - 3\psi_{\infty}^{3,1} - 3\psi_{\infty}^{3,2}.
\end{aligned}$$

On the other hand A_W^* can be described simply as follows:

$$\begin{aligned} A_W^1 &:= \langle \sigma_1, \sigma_2, \sigma_3 \rangle_{\mathbb{C}}, \\ A_W^2 &:= \langle \psi_P^{1,2}, \psi_P^{2,3}, \psi_P^{3,1} : \psi_P^{1,2} + \psi_P^{2,3} + \psi_P^{3,1} = 0 \rangle_{\mathbb{C}}, \\ \sigma_1 \wedge \sigma_2 &= 3\psi_P^{1,2}, \\ \sigma_1 \wedge \sigma_3 &= -3\psi_P^{3,1}, \\ \sigma_2 \wedge \sigma_3 &= 9\psi_P^{2,3}. \end{aligned}$$

Note that in A_W^2 we have eliminated the forms coming from self intersections $(\psi_Q^{1,1'})$, intersections at infinity $(\psi_\infty^{*,*})$ and genus of the components $(\xi_1$ and $\bar{\xi}_1)$.

Also note that $\mathcal{R}_1(\mathcal{C}) := \{\omega = (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1 + 3\lambda_2 + 3\lambda_3 = 0\}$, that is $\mathcal{R}_1(\mathcal{C}) := \{(-3(t+s), t, s)\}$, which corresponds to the fact that $(3\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ form a pencil of cubics.

7.3. Coordinate components of $\mathcal{R}_i(\mathcal{C})$. Consider a nodal cubic \mathcal{C}_3 and a smooth conic \mathcal{C}_2 passing through its node P and intersecting \mathcal{C}_3 at another point Q with multiplicity 4. Consider the line \mathcal{C}_1 joining Q and an inflexion point R of \mathcal{C}_3 , \mathcal{C}_4 joining P and Q , \mathcal{C}_5 joining R and P , and \mathcal{C}_6 the tangent line to \mathcal{C}_3 at R . Define $\{S\} := \mathcal{C}_4 \cap \mathcal{C}_6$, and $\{T_1, T_2\} := \mathcal{C}_2 \cap \mathcal{C}_6$. Consider $\mathcal{C} := \mathcal{C}_1 \cup \dots \cup \mathcal{C}_6$. The following is a description of A_W^* using a basis as in Remark 4.3:

$$(22) \quad \begin{aligned} A_W^1 &:= \langle \sigma_1, \dots, \sigma_6 \rangle_{\mathbb{C}}, \\ A_W^2 &:= \langle \psi_P^{2,3}, \psi_P^{2,3'}, \psi_P^{2,4}, \psi_P^{2,5}, \psi_Q^{1,2}, \psi_Q^{1,3}, \psi_Q^{1,4}, \psi_R^{1,3}, \psi_R^{1,5}, \psi_R^{1,6}, \psi_S^{4,6}, \psi_{T_1}^{2,6}, \psi_{T_2}^{2,6} \rangle_{\mathbb{C}}, \\ \sigma_{1,2} &= 2\psi_Q^{1,2}, & \sigma_{1,3} &= 2\psi_Q^{1,3} + \psi_R^{1,3}, \\ \sigma_{1,k} &= \psi_Q^{1,k} - \psi_Q^{1,k'}, k = 4, 5, 6, \text{ where } * = \mathcal{C}_1 \cap \mathcal{C}_k, & \sigma_{2,3} &= 4\psi_Q^{1,3} - 4\psi_Q^{1,2} + \psi_P^{2,3} + \psi_P^{2,3'}, \\ \sigma_{2,4} &= \psi_Q^{1,4} - \psi_Q^{1,2} + \psi_P^{2,4}, & \sigma_{2,5} &= 2\psi_P^{2,5}, \\ \sigma_{2,6} &= \psi_{T_1}^{2,6} + \psi_{T_2}^{2,6}, & \sigma_{3,4} &= \psi_Q^{1,4} - \psi_Q^{1,3} + 2\psi_P^{2,4} - \psi_P^{2,3} - \psi_P^{2,3'}, \\ \sigma_{3,5} &= \psi_R^{1,5} - \psi_R^{1,3} + 2\psi_P^{2,5} - \psi_P^{2,3} - \psi_P^{2,3'}, & \sigma_{3,6} &= 3\psi_R^{1,6} - 3\psi_R^{1,3}, \\ \sigma_{4,5} &= \psi_P^{2,5} - \psi_P^{2,4}, & \sigma_{4,6} &= \psi_S^{4,6}, \\ \sigma_{5,6} &= \psi_R^{1,6} - \psi_R^{1,5}, \end{aligned}$$

where $\sigma_{i,j}$ denotes $\sigma_i \wedge \sigma_j$. The following matrix represents the map $A_W^1 \xrightarrow{\wedge \omega} A_W^2$ with respect to the bases shown in (22) and $\omega = (t_1, t_2, t_3, t_4, t_5)$:

$$M_\omega := \begin{bmatrix} 0 & t_3 & -t_2 - t_5 - t_4 & t_3 & t_3 & 0 \\ 0 & t_3 & -t_2 - t_5 - t_4 & t_3 & t_3 & 0 \\ 0 & t_4 & 2t_4 & -2t_3 - t_2 - t_5 & t_4 & 0 \\ 0 & 2t_5 & 2t_5 & t_5 & -2t_2 - t_4 - 2t_3 & 0 \\ 2t_2 & -4t_3 - 2t_1 - t_4 & 4t_2 & t_2 & 0 & 0 \\ 2t_3 & 4t_3 & -2t_1 - 4t_2 - t_4 & t_3 & 0 & 0 \\ t_4 & t_4 & t_4 & -t_1 - t_3 - t_2 & 0 & 0 \\ t_3 & 0 & -3t_6 - t_1 - t_5 & 0 & t_3 & 3t_3 \\ t_5 & 0 & t_5 & 0 & -t_1 - t_3 - t_6 & t_5 \\ t_6 & 0 & 3t_6 & 0 & t_6 & -t_1 - t_5 - 3t_3 \\ 0 & 0 & 0 & t_6 & 0 & -t_4 \\ 0 & t_6 & 0 & 0 & 0 & -t_2 \\ 0 & t_6 & 0 & 0 & 0 & -t_2 \end{bmatrix}.$$

Its resonance variety $\mathcal{R}_1(\mathcal{C})$ consists of five linear two-dimensional coordinate components:

$$\begin{aligned} &\{(2t, s, -t-s, 0, t, s)\} \cup \{(-(t+s), 0, t, 2s, -2(t+s), s)\} \cup \{(t, s, 0, -2(t+s), t, 0)\} \cup \\ &\quad \cup \{(-(t+s), t, s, -2(t+s), t, 0)\} \cup \{(-(t+s), 0, 0, 0, t, s)\}, \end{aligned}$$

corresponding to the pencils:

$$(\mathcal{C}_1^2 \mathcal{C}_5, \mathcal{C}_2 \mathcal{C}_6, \mathcal{C}_3), (\mathcal{C}_1 \mathcal{C}_5^2, \mathcal{C}_4^2 \mathcal{C}_6, \mathcal{C}_3), (\mathcal{C}_1 \mathcal{C}_5, \mathcal{C}_2, \mathcal{C}_4^2), (\mathcal{C}_1 \mathcal{C}_4^2, \mathcal{C}_2 \mathcal{C}_5, \mathcal{C}_3), \text{ and } (\mathcal{C}_1, \mathcal{C}_5, \mathcal{C}_6).$$

The remaining $\mathcal{R}_i(\mathcal{C})$, $i \geq 2$ are trivial.

7.4. Explicit computations on the cohomology ring in the non-rational case. We will present a simple example of a non-rational arrangement of curves in order to show how to compute the forms described in §3. Let $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, where $\mathcal{C}_0 := \{x + y + z = 0\}$, $\mathcal{C}_1 := \{y - z = 0\}$, $\mathcal{C}_2 := \{xy + xz + yz = 0\}$, and $\mathcal{C}_3 := \{x^2(y + z) + y^2(x + z) + z^2(x + y) = 0\}$. In this case, for simplicity it is more convenient to consider the line at infinity \mathcal{C}_0 with an equation different from $\{z = 0\}$. Consider ξ a primitive third root of unity (a root of $t^2 + t + 1 = 0$) and denote $\mathcal{C}_0 \cap \mathcal{C}_1 = \{P_{01} = [-2 : 1 : 1]\}$, $\mathcal{C}_0 \cap \mathcal{C}_2 = \{R_1 = [-\bar{\xi} - 1 : \bar{\xi} : 1], R_2 = [-1 - \xi : \xi : 1]\}$, $\mathcal{C}_0 \cap \mathcal{C}_3 = \{Q_1 = [0 : 1 : -1], Q_2 = [-1 : 0 : 1], Q_3 = [-1 : 1 : 0]\}$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \{P_1, P_{12} = [1 : -2 : -2]\}$, $\mathcal{C}_1 \cap \mathcal{C}_3 = \{P_1, P_{13} = [\xi : 1 : 1], \bar{P}_{13} = [\bar{\xi} : 1 : 1]\}$, $\mathcal{C}_2 \cap \mathcal{C}_3 = \{P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0], P_3 = [0 : 0 : 1]\}$,

Since all the local branches of the irreducible components at any singular point are irreducible, we will denote by $\psi_P^{i,j}$ the 2-form associated with the singular point P and the local branches at P of \mathcal{C}_i and \mathcal{C}_j . For example, in order to compute $\psi_{P_1}^{2,3} = \varphi_{P_1}^{2,3} \frac{\omega}{C_0 C_2 C_3}$, one needs a section of $H^0(\mathbb{P}^2, \mathcal{I}_{P_1}^{2,3}(3))$. Note that

$$\left(\mathcal{I}_{P_1}^{2,3}\right)_P := \begin{cases} \{\varphi \in \mathcal{O}_P \mid \mathcal{T}_P(C_{2,3})|_{\varphi} \geq \mathcal{T}_{\mathbb{A}_3}\} = \mathfrak{m}_P & \text{if } P = P_1, \\ \{\varphi \in \mathcal{O}_P \mid \mathcal{T}_P(C_{2,3})|_{\varphi} \geq \mathcal{T}_{\mathbb{A}_3}^n\} & \text{if } P = P_2, P_3, \\ \mathcal{O}_P & \text{otherwise,} \end{cases}$$

where Figure 4 describes the local conditions at the tacnodes P_i .

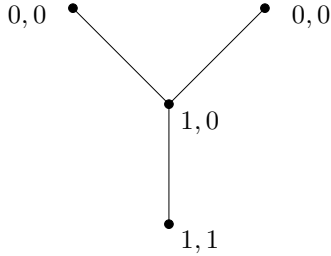


FIGURE 4. Description of $\mathcal{T}_{\mathbb{A}_3}^n$, and $\mathcal{T}_{\mathbb{A}_3}$ respectively.

Therefore $\varphi_{P_1}^{2,3}$ is the equation of a cubic $\alpha(xz + x^2 + (1 - \xi)xy + yz)z + \beta C_0 C_2$. In order to obtain a normal form one has to require the different residues of $\psi_{P_1}^{2,3}$ at P_1 and at an exceptional divisor E joining δ_2 and δ_3 to be equal to ± 1 . It is a simple computation that

$$\text{Res}_{P_1}^{[2]} \psi_{P_1}^{2,3} = \frac{\alpha}{3}$$

and that

$$\left({}^1 R_0^{[1]} \psi_{P_1}^{2,3}\right)_E = \frac{1}{3}(\beta - \xi).$$

Since $\left({}^1 R_0^{[1]} \frac{d\mathcal{C}_2}{\mathcal{C}_2} \wedge \frac{d\mathcal{C}_3}{\mathcal{C}_3}\right)_E = -\frac{2}{3}$ and $(\delta_2, \delta_3)_{P_1} = 2$, one concludes that

$$\varphi_{P_1}^{2,3} = 3(xz + x^2 + (1 - \xi)xy + yz)z + (\xi - 1)C_0 C_2.$$

Analogously one can proceed with $\psi_{P_1}^{1,2} = \varphi_{P_1}^{1,2} \frac{\omega}{C_0 C_1 C_2}$ and $\psi_{P_1}^{3,1} = \varphi_{P_1}^{3,1} \frac{\omega}{C_0 C_3 C_1}$ obtaining $\varphi_{P_1}^{1,2} := 2x - \xi y + (1 + \xi)z$ and $\varphi_{P_1}^{1,3} := 2(x^2 + xz + 2yz - y^2) + C_0 C_1$.

Note that $\varphi_{P_1}^{1,2} C_3 + \varphi_{P_1}^{2,3} C_1 - \varphi_{P_1}^{1,3} C_2 = 0$ and hence $\psi_{P_1}^{1,2} + \psi_{P_1}^{2,3} + \psi_{P_1}^{3,1} = 0$ (Theorem 3.27(R3)).

The following list describes the polynomials $\varphi_P^{i,j}$ for the generating 2-forms $\psi_P^{i,j} := \varphi_P^{i,j} \frac{\omega}{C_0 C_i C_j}$:

$$\begin{aligned}\varphi_{P_1}^{2,3} &:= (\xi + 2)(zx^2 + \xi yx^2 + xz^2 + \xi y^2x + z^2y + \xi zy^2), \\ \varphi_{P_2}^{2,3} &:= (\xi + 2)(y^2x + xz^2 + yx^2 + zx^2 + z^2y + (1 - \xi)zxy + \xi zy^2), \\ \varphi_{P_3}^{2,3} &:= (\xi - 1)(y^2x + xz^2 + yx^2 + zx^2 + y^2z + (\xi + 2)zxy - (1 + \xi)z^2y), \\ \varphi_{P_1}^{1,2} &:= 2x - \xi y + (1 + \xi)z, \\ \varphi_{P_{12}}^{1,2} &:= (\xi - 1)((\xi + 1)y + z), \\ \varphi_{P_1}^{1,3} &:= 2x^2 + xz + xy - z^2 + 4yz - y^2, \\ \varphi_{P_{13}}^{1,3} &:= -(\xi + 2)(xz + xy + z^2 + 2\xi zy + y^2), \\ \varphi_{P_{13}}^{1,3} &:= (\xi - 1)(xz + xy + z^2 - 2(\xi + 1)yz + y^2).\end{aligned}$$

Finally we also describe the polynomials $\varphi_\infty^{i,k}$ for the generating 2-forms $\psi_\infty^{i,k} := \varphi_\infty^{i,k} \frac{\omega}{C_0 C_i}$:

$$\begin{aligned}\varphi_\infty^{3,Q_2} &:= -3(x + y), \\ \varphi_\infty^{3,Q_3} &:= 3(x + z), \\ \varphi_\infty^{2,R_2} &:= -(2\xi + 1).\end{aligned}$$

One can then easily verify (17), that is,

$$\text{Jac}(C_2, C_3, C_0) = 2\varphi_{P_1}^{2,3} + 2\varphi_{P_2}^{2,3} + 2\varphi_{P_3}^{2,3} + 3\varphi_\infty^{2,R_2} C_3 - 2\varphi_\infty^{3,Q_2} C_2 - 2\varphi_\infty^{3,Q_3} C_2,$$

and hence

$$\sigma_2 \wedge \sigma_3 = 2\psi_{P_1}^{2,3} + 2\psi_{P_2}^{2,3} + 2\psi_{P_3}^{2,3} + 3\psi_\infty^{2,R_2} - 2\psi_\infty^{3,Q_2} - 2\psi_\infty^{3,Q_3},$$

which corresponds to Theorem 3.27(R2).

The same can be checked for $\text{Jac}(C_1, C_2, C_0)$ and $\text{Jac}(C_1, C_3, C_0)$.

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